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Associativity of products of existence varieties of regular semigroups[1]

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Abstract

An e-variety is a class of regular semigroups that is closed under the formation of direct products, homomorphic images and regular subsemigroups. In a previous paper, the authors established that, for any nongroup e-variety $\mathscr{V}$, the e-variety $\mathscr{LG} \circ \mathscr{V}$, where $\circ$ denotes the Mal'cev product within the class of regular semigroups and $\mathscr{LG}$ denotes the e-variety of left groups, is actually equal to the e-variety $\mathscr{G} \otimes \mathscr{V}$ generated by all wreath products of the form $G \otimes T$, where $G \in \mathscr{G}$, the e-variety of all groups, and $T \in \mathscr{V}$. It was also shown that if $\mathscr{LZ}$ denotes the e-variety of left zero semigroups and $\mathscr{S}$ the e-variety of all semilattices, then $\mathscr{LZ} \circ \mathscr{V}$ is equal to the e-variety $\mathscr{S} \otimes^* \mathscr{V}$ generated by certain subsemigroups of the wreath products of the form $S \otimes T$, where $S \in \mathscr{S}$ and $T \in \mathscr{V}$. In this paper, the e-varieties generated by the regular parts of the wreath products of the form $\mathscr{RZ} \otimes \mathscr{V}$, $\mathscr{RB} \otimes \mathscr{V}$ and $\mathscr{CS} \otimes \mathscr{V}$, where $\mathscr{RZ}$, $\mathscr{RB}$ and $\mathscr{CS}$ denote the e-varieties of right zero semigroups, rectangular bands and completely simple semigroups respectively, are studied and are found, in general, to fall far short of the corresponding Mal'cev products. An important tool is the associativity of the wreath product of e-varieties under certain conditions and a substantial part of the paper is devoted to this issue.

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1. Introduction

The class of all regular semigroups is not closed under the operation of taking subsemigroups and therefore cannot be considered as a variety of algebras. However, some of the ideas and techniques of universal algebra that have been so potent in the study of other classes were made available for the investigation of regular semigroups

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through the introduction of the concept of an e-variety, that is a class of regular semigroups that is closed under the formation of direct products, homomorphic images and regular subsemigroups. This concept was introduced by Hall [7] and Kaďourek and Szendrei [14]. The set of all e-varieties forms a complete lattice $\mathcal{L}_{ev}(\mathcal{RS})$ [7]. In [24] the authors introduced certain complete congruences on this lattice. Associated with each of these complete congruences is the operator that maps each element to the maximum element in its class. These operators assume the form

$$\mathcal{V} \to \mathcal{V}^{T_l} = \mathcal{LG} \circ \mathcal{V}, \quad \mathcal{V} \to \mathcal{V}^{T_r} = \mathcal{RG} \circ \mathcal{V}, \quad \mathcal{V} \to \mathcal{V}^{M} = \mathcal{CS} \circ \mathcal{V},$$
$$\mathcal{V} \to \mathcal{V}^{K_l} = \mathcal{LZ} \circ \mathcal{V}, \quad \mathcal{V} \to \mathcal{V}^{K_r} = \mathcal{RZ} \circ \mathcal{V}, \quad \mathcal{V} \to \mathcal{V}^{K} = \mathcal{RB} \circ \mathcal{V}.$$

In [25], the authors provided alternative characterizations of the e-varieties $\mathcal{LG} \circ \mathcal{V}$ and $\mathcal{LZ} \circ \mathcal{V}$ in terms of wreath products. It so happens that the wreath product of a group with a regular semigroup is again a regular semigroup so that it is possible to define $\mathcal{G} \otimes \mathcal{V}$ to be the e-variety of regular semigroups generated by all wreath products of groups with elements from $\mathcal{V}$. With this construction to hand it was shown in [25] that

$$\mathcal{V}^{T_l} = \mathcal{G} \otimes \mathcal{V}.$$

A similar result was obtained for $\mathcal{V}^{K_l}$ in terms of a modified wreath product $\mathcal{S} \otimes^* \mathcal{V}$, where $\mathcal{S}$ is the e-variety of all semilattices, as well as other results characterizing Mal'cev products in terms of wreath products. For an interesting study of the application of wreath products to the description of e-free objects in e-varieties, see [12].

In order to extend the investigations of the relationships between Mal'cev products of the form $\mathcal{U} \circ \mathcal{V}$ and wreath products of the form $\mathcal{U} \otimes \mathcal{V}$, for e-varieties $\mathcal{U}$ and $\mathcal{V}$ where the e-variety $\mathcal{U}$ assumes a more general form than in the cases referred to above, it is natural to consider such e-varieties as the e-variety $\mathcal{CS}$ of completely simple semigroups and the e-variety $\mathcal{RB}$ of rectangular bands. In general, the wreath product $S \otimes T$ of two regular semigroups need not be regular. However, Jones and Trotter [12] have shown that the set $\mathrm{Reg}(S \otimes T)$ of regular elements in $S \otimes T$ will form a regular subsemigroup provided either S or T belongs to $\mathcal{CS}$, the e-variety of completely simple semigroups. This makes it possible to define $\mathcal{U} \otimes \mathcal{V}$, whenever $\mathcal{U}$ or $\mathcal{V}$ is contained in $\mathcal{CS}$, to be the e-variety generated by the regular parts of the wreath products $S \otimes T$, for all $S \in \mathcal{U}$, $T \in \mathcal{V}$.

However, since $\mathcal{CS} = \mathcal{G} \otimes \mathcal{RZ}$, we quickly find ourselves concerned with the issue of associativity of the wreath product of e-varieties: $(\mathcal{U} \otimes \mathcal{V}) \otimes \mathcal{W} = \mathcal{U} \otimes (\mathcal{V} \otimes \mathcal{W})$. As a consequence, after reviewing some required background in Section 2 and dealing with some preliminary technical observations in Section 3, the main results in Section 4 identify certain important circumstances under which the associative law $(\mathcal{U} \otimes \mathcal{V}) \otimes \mathcal{W} = \mathcal{U} \otimes (\mathcal{V} \otimes \mathcal{W})$ will hold for e-varieties $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$. In particular, it is shown that associativity prevails in the following circumstances:

(i) $\mathcal{U}, \mathcal{W} \subseteq \mathcal{G}$ and $\mathcal{V} \subseteq \mathcal{ES}$, or

(ii) $\mathcal{V}, \mathcal{W} \subseteq \mathcal{G}$ and $\mathcal{U} \in \mathcal{L}_{ev}(\mathcal{IL}) \cup \mathcal{L}_{ev}(\mathcal{ES})$, or

(iii) $\mathcal{U}, \mathcal{V} \subseteq \mathcal{G}$ and $\mathcal{W}$ is monoidal, that is, such that $S^1 \in \mathcal{W}$ for all $S \in \mathcal{W}$.

In addition, it is shown that $\mathscr{S} \otimes^* (\mathscr{U} \otimes \mathscr{V}) = (\mathscr{S} \otimes^* \mathscr{U}) \otimes \mathscr{V}$, provided $\mathscr{V} \subseteq \mathscr{G}$, where $\mathscr{S} \otimes^* \mathscr{W}$ denotes the e-variety generated by all products, involving a modified version of the wreath product, of elements from $\mathscr{S}$ and any e-variety $\mathscr{W}$ of E-solid regular semigroups.

For any e-variety $\mathscr{U}$, we denote by $\mathscr{U}C_\infty$ the e-variety consisting of all regular semigroups S for which the least full self-conjugate subsemigroup $C_\infty(S)$ of S lies in $\mathscr{U}$. It was essentially established by Jones and Trotter [12] that, for any e-subvariety $\mathscr{U}$ of $\mathscr{O}$, $\mathscr{U} \otimes \mathscr{G} = \mathscr{U} * \mathscr{G} = \mathscr{U}C_\infty = \mathscr{U}C$. Section 5 begins with an example to show that the equality $\mathscr{U} \otimes \mathscr{G} = \mathscr{U}C_\infty$ does not hold in general. Indeed, we provide an infinite family of such examples. This is followed by the first main result of the section which establishes that for certain e-varieties $\mathscr{U}$, if equality does prevail then it will also prevail when $\mathscr{U}$ is replaced by its image under the operators referred to above. Explicitly, if $\mathscr{U}$ is an e-subvariety of $\mathscr{ES}$ or $\mathscr{I}L$ and $\mathscr{U}C_\infty = \mathscr{U} \otimes \mathscr{G}$, then we also have $(\mathscr{U}^P)C_\infty = (\mathscr{U}^P) \otimes \mathscr{G}$ for any $P \in \{T_l, T_r, K_l, K_r\}$.

For any variety of groups $\mathscr{K}$, we denote by $\hat{\mathscr{K}}$ the largest variety of inverse semigroups having E-unitary covers over $\mathscr{K}$. The second main result of Section 5 establishes the equalities

$$\mathscr{CS} \circ \hat{\mathscr{K}} = \langle \mathscr{CR} \circ \mathscr{K} \rangle_{er} = \mathscr{CR} \otimes \mathscr{K}, \qquad \mathscr{RB} \circ \hat{\mathscr{K}} = \langle \mathscr{B} \circ \mathscr{K} \rangle_{er} = \mathscr{B} \otimes \mathscr{K}.$$

The first part of Section 6 is devoted to the study of e-varieties of the form $\mathscr{RZ} \otimes \mathscr{V}$. In particular, it is shown that for any e-variety $\mathscr{V}$, $\mathscr{RZ} \otimes \mathscr{V}^{K_r} = \mathscr{V}^{K_r}$ so that for e-varieties such as $\mathscr{B}, \mathscr{CS}, \mathscr{O}, \mathscr{ES}$, for instance, $\mathscr{RZ} \otimes \mathscr{V} = \mathscr{V}$. For any e-subvariety $\mathscr{V}$ of $\mathscr{CR}$ or $\mathscr{O}L$, $\mathscr{RZ} \otimes \mathscr{V} = \mathscr{RZ} \vee \mathscr{V}$. In the main result of this section, it is shown that if $\mathscr{V}$ is a monoidal e-variety such that $\mathscr{RZ} \otimes \mathscr{V} = \mathscr{V}$ then for any e-subvariety $\mathscr{U}$ of $\mathscr{CS}$,

$$\mathscr{U} \otimes \mathscr{V} = \begin{cases} (\mathscr{U} \cap \mathscr{G}) \otimes \mathscr{V} & \text{if } \ \mathscr{U} \cap \mathscr{G} \neq \mathscr{T}, \\ (\mathscr{U} \cap \mathscr{LZ}) \vee \mathscr{V} & \text{otherwise.} \end{cases}$$

These results have various applications to special cases. For example, $\mathscr{CS} \otimes \mathscr{S} = \mathscr{LG} \circ (\mathscr{RZ} \vee \mathscr{S})$, $\mathscr{CS} \otimes \mathscr{B} = \mathscr{LG} \circ \mathscr{B}$, $\mathscr{CS} \otimes \mathscr{I} = \mathscr{LG} \circ (\mathscr{RZ} \vee \mathscr{I})$, $\mathscr{CS} \otimes \mathscr{O} = \mathscr{LG} \circ \mathscr{O}$.

One surprising outcome from this work is that less is gained by consideration of products of the more general forms $\mathscr{RB} \otimes \mathscr{V}$ and $\mathscr{CS} \otimes \mathscr{V}$ than might have been expected. From the results referred to above, we find that for any e-subvariety $\mathscr{V}$ of $\mathscr{CR}$ or $\mathscr{O}L$,

$$\mathscr{RB} \otimes \mathscr{V} = \mathscr{RB} \vee \mathscr{V}$$

while in the final result of the paper it is shown that for any e-variety $\mathscr{V}$ with $\mathscr{RZ} \subseteq \mathscr{V} \subseteq \mathscr{CR}$,

$$\mathscr{CS} \otimes \mathscr{V} = \mathscr{V}^{T_l}.$$

2. Preliminaries

In general, we use the notation and terminology of Howie [9] and Petrich [21]. For background on varieties of algebras, consult Grätzer [4].

Let A and B be nonempty sets. We denote by A^B the set of all mappings from B to A. If $\theta \in A^B$, then $\bar{\theta}$ denotes the equivalence relation on B induced by θ. The equality relation on any set is denoted by ι.

For any semigroup S, we denote by S^1 the smallest monoid containing S; that is, S itself if S is already a monoid, and $S \cup \{1\}$ otherwise. The set of all regular elements of S is denoted by Reg(S). In general, Reg(S) is not a subsemigroup of S. Hall [6, Result 2] pointed out the following useful result.

Result 2.1. *For any semigroup S, Reg (S) is a subsemigroup of S if and only if the product of any pair of idempotents is regular, that is, $E(S)^2 \subseteq \mathrm{Reg}(S)$.*

Throughout this paper $\mathscr{RS}$ stands for the class of all regular semigroups. Let $S \in \mathscr{RS}$. Then $E(S)$ denotes the set of all idempotents of S and $C(S)$ denotes the *core* of S, that is, the subsemigroup of S generated by $E(S)$. If $x \in S$, then $V(x)$ is the set of all inverses of x; if x belongs to a subgroup of S, then x^{-1} is the inverse of x in that subgroup and $x^0 = x^{-1}x$. The *sandwich set* of elements x and y of S is $S(x, y) = yV(xy)x$.

Let $S \in \mathscr{RS}$. If θ is an equivalence relation on S, then θ^0 denotes the greatest congruence contained in θ. If θ_1 and θ_2 are equivalence relations on S, then clearly $(\theta_1 \cap \theta_2)^0 = \theta_1{}^0 \cap \theta_2{}^0$.

For any nonempty class $\mathscr{U}$ of regular semigroups we define $\mathbf{H}\mathscr{U}$, $\mathbf{S_r}\mathscr{U}$ and $\mathbf{P}\mathscr{U}$ as follows: $\mathbf{H}\mathscr{U}$ is the class of all (regular) semigroups that are homomorphic images of semigroups in $\mathscr{U}$; $\mathbf{S_r}\mathscr{U}$ is the class of all *regular* subsemigroups of semigroups in $\mathscr{U}$; $\mathbf{P}\mathscr{U}$ is the class of all direct products of semigroups in $\mathscr{U}$.

As in [7], a class $\mathscr{U}$ of regular semigroups is called an *existence variety*, or e-variety, if $\mathbf{H}\mathscr{U} \subseteq \mathscr{U}$, $\mathbf{S_r}\mathscr{U} \subseteq \mathscr{U}$ and $\mathbf{P}\mathscr{U} \subseteq \mathscr{U}$. The class of all e-varieties of regular semigroups forms a complete lattice under inclusion, and is denoted by $\mathscr{L}_{ev}(\mathscr{RS})$.

Let $\mathscr{A}$ be a subclass of $\mathscr{RS}$. We denote by $\langle \mathscr{A} \rangle_{ev}$ the e-variety of regular semigroups *generated* by $\mathscr{A}$. Furthermore, the lattice of e-subvarieties of an e-variety $\mathscr{U}$ shall be denoted by $\mathscr{L}_{ev}(\mathscr{U})$.

The following e-subvarieties of $\mathscr{RS}$ will figure prominently in our discussions:

$\mathscr{T}$ – the e-variety of all trivial semigroups;

$\mathscr{S}$ – the e-variety of all semilattices;

$\mathscr{LZ}$ – the e-variety of all left zero semigroups;

$\mathscr{RZ}$ – the e-variety of all right zero semigroups;

$\mathscr{RB}$ – the e-variety of all rectangular bands;

$\mathscr{B}$ – the e-variety of all bands;

$\mathscr{G}$ – the e-variety of all groups;

$\mathscr{LG}$ – the e-variety of all left groups;

$\mathscr{RG}$ – the e-variety of all right groups;
$\mathscr{AG}$ – the e-variety of all abelian groups;
$\mathscr{ReG}$ – the e-variety of all rectangular groups;
$\mathscr{CS}$ – the e-variety of all completely simple semigroups;
$\mathscr{CR}$ – the e-variety of all completely regular semigroups;
$\mathscr{I}$ – the e-variety of all inverse semigroups;
$\mathscr{O}$ – the e-variety of all orthodox semigroups.

Given e-varieties $\mathscr{U}$ and $\mathscr{V}$ of regular semigroups, their *Mal'cev product* $\mathscr{U} \circ \mathscr{V}$ consists of those regular semigroups S for which there exists a congruence ρ on S such that $S/\rho \in \mathscr{V}$ and $e\rho \in \mathscr{U}$ for all $e \in E(S)$. In general $\mathscr{U} \circ \mathscr{V}$ is not an e-variety (see [10]). But for certain choices of e-varieties $\mathscr{U}$ and $\mathscr{V}$, $\mathscr{U} \circ \mathscr{V}$ is indeed an e-variety.

Result 2.2 (Reilly and Zhang [24, Section 5]). *If* $\mathscr{U} \in \{\mathscr{LZ}, \mathscr{RZ}, \mathscr{RB}, \mathscr{G}, \mathscr{LG}, \mathscr{RG}, \mathscr{CS}\}$ *and* $\mathscr{V} \in \mathscr{L}_{er}(\mathscr{RS})$, *then* $\mathscr{U} \circ \mathscr{V}$ *is again an e-variety.*

The following operators on $\mathscr{L}_{er}(\mathscr{RS})$ were introduced by the authors in [24]: for any $\mathscr{U} \in \mathscr{L}_{er}(\mathscr{RS})$,

$$\mathscr{U}^{T_\ell} = \mathscr{LG} \circ \mathscr{U}, \qquad \mathscr{U}^{T_r} = \mathscr{RG} \circ \mathscr{U},$$
$$\mathscr{U}^{K_\ell} = \mathscr{LZ} \circ \mathscr{U}, \qquad \mathscr{U}^{K_r} = \mathscr{RZ} \circ \mathscr{U}.$$

It is easily verified that T_ℓ, T_r, K_ℓ and K_r are closure operators on $\mathscr{L}_{er}(\mathscr{RS})$. We also have the following useful characterizations.

Result 2.3 (Reilly and Zhang [24, Section 5]). *Let* $\mathscr{U} \in \mathscr{L}_{er}(\mathscr{RS})$. *Then*
(i) $\mathscr{U}^{K_\ell} = \{S \in \mathscr{RS} \mid S/(\pi \cap \mathscr{L})^0 \in \mathscr{U}\}$,
(ii) $\mathscr{U}^{K_r} = \{S \in \mathscr{RS} \mid S/(\pi \cap \mathscr{R}) \in \mathscr{U}\}$,
(iii) $\mathscr{U}^{T_\ell} = \{S \in \mathscr{RS} \mid S/\mathscr{L}^0 \in \mathscr{U}\}$,
(iv) $\mathscr{U}^{T_r} = \{S \in \mathscr{RS} \mid S/\mathscr{R}^0 \in \mathscr{U}\}$,
where $\pi = \{(a,b) \in S \times S \mid V(a) = V(b)\}$.

The next result will be used frequently in Section 4. The first part is specialized from the proof of [17, Ch. 32, Theorem 8], and the second part can be easily verified by using the same type of argument as in the proof of [32, Corollary 6.5].

Result 2.4. *For any* $\mathscr{U}$, $\mathscr{V}$ *and* $\mathscr{W} \in \mathscr{L}_{er}(\mathscr{RS})$, *we have*
(a) $\mathscr{U} \circ (\mathscr{V} \circ \mathscr{W}) \subseteq (\mathscr{U} \circ \mathscr{V}) \circ \mathscr{W}$ *and*
(b) *if* $\mathscr{U}, \mathscr{V} \subseteq \mathscr{LG}$, *then* $\langle \mathscr{U} \circ \langle \mathscr{V} \circ \mathscr{W} \rangle_{er} \rangle_{er} \subseteq \langle \langle \mathscr{U} \circ \mathscr{V} \rangle_{er} \circ \mathscr{W} \rangle_{er}$.

The operators C and L are defined on $\mathscr{L}_{er}(\mathscr{RS})$ as follows:

$$\mathscr{U}C = \{S \in \mathscr{RS} \mid C(S) \in \mathscr{U}\}$$

and

$$\mathscr{U}L = \{S \in \mathscr{RS} \mid eSe \in \mathscr{U} \text{ for all } e \in E(S)\}.$$

These two operators were introduced by Hall in [7], where the notation $\mathscr{U}^{ig}$ and $\mathscr{U}^{loc}$ are used and it is shown that $\mathscr{U}C, \mathscr{U}L \in \mathscr{L}_{ev}(\mathscr{RS})$ and that both C and L are closure operators on $\mathscr{L}_{ev}(\mathscr{RS})$. The commutativity of C and L with other operators was considered in [26].

A regular semigroup S is called *E-solid* if for all idempotents $e, f, g \in S$ such that $e\mathscr{L}f\mathscr{R}g$ there exists an idempotent $h \in S$ such that $e\mathscr{R}h\mathscr{L}g$. We denote by $\mathscr{ES}$ the class of all E-solid regular semigroups. It is well known that $\mathscr{ES}$ is an e-variety of regular semigroups and $\mathscr{ES} = (\mathscr{CR})C = \mathscr{CS} \circ \mathscr{I}$ (see [5, 8, 31]). The e-varieties $\mathscr{ES}$ and $\mathscr{I}L$ will play an important role in this paper.

3. Products of *e*-varieties

Recall that a *relational morphism* between two semigroups S and T is a relation $\tau : S \to T$ such that (for example, see [30]):

(i) $s\tau \neq \phi$ for all $s \in S$;

(ii) $(s\tau)(t\tau) \subseteq (st)\tau$ for all $s, t \in S$.

Equivalently, τ is a relation whose *graph*

$$g(\tau) = \{(s,t) \in S \times T \mid t \in s\tau\}$$

is a subsemigroup of $S \times T$ that projects onto S. A relational morphism $\tau : S \to T$ is called *injective* if $s\tau \cap t\tau \neq \emptyset$ implies $s = t$, and is called *surjective* if $t\tau^{-1} = \{s \in S \mid t \in s\tau\} \neq \emptyset$ for all $t \in T$ (see [3, Chs. XI and XII] or [23]).

Let $S, T \in \mathscr{RS}$. A relational morphism $\tau : S \to T$ is called *regular* if $g(\tau)$ is a regular subsemigroup of $S \times T$.

Following Jones and Trotter [12], we say that a regular semigroup S *regularly divides* a regular semigroup T, denoted by $S <_r T$, if there is a regular subsemigroup R of T and an epimorphism of R onto S. The relation $<_r$ in general is not transitive.

Result 3.1 (Reilly and Zhang [25]). *Let $S, T \in \mathscr{RS}$. Then $S <_r T$ if and only if there is an injective regular relational morphism from S to T.*

Let S and T be semigroups. A *left action* of T on S is a map $T \times S \to S, (t,s) \to {}^t s$ satisfying: (i) ${}^{t_1}({}^{t_2}s) = {}^{t_1 t_2}s$ and (ii) $({}^t s_1)({}^t s_2) = {}^t(s_1 s_2)$, for all $t, t_1, t_2 \in T$ and $s, s_1, s_2 \in S$. If T is a monoid, the action is *left unitary* if ${}^1 s = s$ for all $s \in S$. The *semidirect product* $S * T$ of S and T, with respect to this left action, is the set $S \times T$ with the operation $(s_1, t_1)(s_2, t_2) = (s_1 {}^{t_1}s_2, t_1 t_2)$. The dual concepts are the *right action* of T on S and the *reverse semidirect product* $S *_r T$.

A special case is the *wreath product* of arbitrary semigroups S and T. For each $f : T^1 \to S$ and $t \in T$, let ${}^t f : T^1 \to S$ be defined by $x({}^t f) = (xt)f$. Then the map

$(t,f) \to {}^t f$ defines a left action (left unitary whenever T is a monoid) of T on S^{T^1}, the *wreath product* $S \otimes T$ of S and T is the semidirect product $S^{T^1} * T$ of S^{T^1} and T with respect to this action. The dual concept is the *reverse wreath product* $S \otimes_r T$.

The regularity of semidirect and wreath products has been studied by Nico [19] and Skornyakov [27]. The following general result was proved by Jones and Trotter.

Result 3.2 (Jones and Trotter [12]). *Let $S, T \in \mathcal{RS}$. For any left action of T on S, we have*

(i) *if either $S \in \mathcal{LG}$ or $T \in \mathcal{G}$, then $S * T$ is regular;*

(ii) *if either $S \in \mathcal{CS}$ or $T \in \mathcal{CS}$, then* $\mathrm{Reg}(S * T)$ *is a regular subsemigroup of $S * T$.*

Clearly, the analogues of Result 3.2 for wreath products, reverse semidirect and wreath products also hold.

Let $\mathcal{U}, \mathcal{V} \in \mathcal{L}_{er}(\mathcal{RS})$ be such that either $\mathcal{U} \subseteq \mathcal{CS}$ or $\mathcal{V} \subseteq \mathcal{CS}$. Following Jones and Trotter [12], we define the *semidirect product* $\mathcal{U} * \mathcal{V}$ of $\mathcal{U}$ and $\mathcal{V}$ to be

$$\mathcal{U} * V = \left\langle \mathrm{Reg}(S * T) \;\middle|\; \begin{array}{c} S \in \mathcal{U}, T \in \mathcal{V} \text{ and whenever} \\ T \text{ is a monoid its action on} \\ S \text{ is left unitary} \end{array} \right\rangle_{er}$$

and the *wreath product* $\mathcal{U} \otimes \mathcal{V}$ of $\mathcal{U}$ and $\mathcal{V}$ is

$$\mathcal{U} \otimes V = \langle \mathrm{Reg}(S \otimes T) \mid S \in \mathcal{U}, T \in \mathcal{V} \rangle_{er}.$$

Similarly, their reverse semidirect product and reverse wreath product are denoted by $\mathcal{U} *_r \mathcal{V}$ and $\mathcal{U} \otimes_r \mathcal{V}$. For the case where $\mathcal{U} \subseteq \mathcal{G}$, these products were introduced and studied by the authors in [25].

The following results will be needed in the sequel.

Result 3.3 (Jones and Trotter [12]). *Let $\mathcal{U}, \mathcal{V} \in \mathcal{L}_{er}(\mathcal{RS})$ be such that either $\mathcal{U} \subseteq \mathcal{CS}$ or $\mathcal{V} \subseteq \mathcal{CS}$. Then*

(a) $\mathcal{U} * \mathcal{V} = \mathcal{U} \otimes \mathcal{V}$;

(b) *if $\mathcal{U} * \mathcal{V}$ is contained within either $\mathcal{ES}$ or $\mathcal{I}L$,*

$$\mathcal{U} * \mathcal{V} = \{S \in \mathcal{RS} \mid S <_r \mathrm{Reg}(R * T) \textit{ for some } R \in \mathcal{U} \textit{ and } T \in \mathcal{V}\}$$
$$= \{S \in \mathcal{RS} \mid S <_r \mathrm{Reg}(R \otimes T) \textit{ for some } R \in \mathcal{U} \textit{ and } T \in \mathcal{V}\}.$$

Result 3.4 (Jones and Trotter [12]). *Let $\mathcal{U}, \mathcal{V} \in \mathcal{L}_{er}(\mathcal{RS})$ with $\mathcal{V} \subseteq \mathcal{CS}$. Then*

(a) *$\mathcal{U} * \mathcal{V} \subseteq \mathcal{I}L$ if and only if $\mathcal{U} \subseteq \mathcal{I}L$, and*

(b) *$\mathcal{U} * \mathcal{V} \subseteq \mathcal{ES}$ if and only if either $\mathcal{U} \subseteq \mathcal{CS}$ or both $\mathcal{U} \subseteq \mathcal{ES}$ and $\mathcal{V} \subseteq \mathcal{LG}$.*

Result 3.5 (Jones and Trotter [12]). *Let $\mathcal{U}, \mathcal{V} \in \mathcal{L}_{er}(\mathcal{RS})$ with $\mathcal{U} \subseteq \mathcal{CS}$. Then*

(a) *$\mathcal{U} * \mathcal{V} \subseteq \mathcal{I}L$ if and only if either $\mathcal{V} \subseteq \mathcal{CS}$ or both $\mathcal{U} \subseteq \mathcal{RB}$ and $\mathcal{V} \subseteq \mathcal{I}L$, and*

(b) *$\mathcal{U} * \mathcal{V} \subseteq \mathcal{ES}$ if and only if $\mathcal{V} \subseteq \mathcal{ES}$.*

Result 3.6 (Reilly and Zhang [25]). *Let* $\mathscr{U} \in \mathscr{L}_{er}(\mathscr{G})\backslash\{\mathscr{T}\}$ *and* $\mathscr{V} \in \mathscr{L}_{er}(\mathscr{RS})$ *be such that* $\mathscr{V} \not\subseteq \mathscr{G}$. *Then* $\mathscr{U} \otimes \mathscr{V} = \langle(\mathscr{L}\mathscr{Z} \vee \mathscr{U}) \circ \mathscr{V}\rangle_{er}$. *In particular,* $\mathscr{V}^{T_l} = \mathscr{G} \otimes \mathscr{V}$.

Lemma 3.7. *Let* $\mathscr{U} \in \mathscr{L}_{er}(\mathscr{RS})$ *and* $\mathscr{V} \in \mathscr{L}_{er}(\mathscr{G})$. *Then*
(a) $\mathscr{U} * \mathscr{V} \subseteq \langle \mathscr{U} \circ \mathscr{V}\rangle_{er}$.
(b) $\mathscr{U} * \mathscr{V} = \mathscr{U} *_r \mathscr{V}$.

Proof. (i) Let $S \in \mathscr{U}$ and $G \in \mathscr{V}$. Then for any left unitary left action of G on S, the projection map $p:(s,g) \to g((s,g) \in \mathrm{Reg}(S * G) = S * G)$ is an epimorphism of $S * G$ onto G such that $1p^{-1} \cong S$, where 1 is the identity of G. It follows that $S * G \in \mathscr{U} \circ \mathscr{V}$ whence $\mathscr{U} * \mathscr{V} \subseteq \langle \mathscr{U} \circ \mathscr{V}\rangle_{er}$.

(ii) This follows immediately from Margolis and Pin [16, Proposition 1.3 and its dual]. □

Lemma 3.8. *Let* $S, T \in \mathscr{RS}$ *be such that* $S <_r T$. *Then*

$$G \otimes S <_r G \otimes T \quad \textit{and} \quad S \otimes G <_r T \otimes G$$

for any $G \in \mathscr{G}$.

Proof. Let $G \in \mathscr{G}$. Since $S <_r T$, then by Result 3.1 there is an injective regular relational morphism $\tau : S \to T$.

(a) $G \otimes S <_r G \otimes T$. For each $(\varphi, s) \in G \otimes S$, we let

$$(\varphi,s)\tau_1 = \left\{ (\phi,t) \in G \otimes T \;\middle|\; \begin{array}{c} t \in s\tau \text{ and } \phi \text{ satisfies} \\ \text{(i) } 1_T\phi = 1_S\varphi; \\ \text{(ii) } x \in y\tau \text{ implies } x\phi = y\varphi \end{array} \right\}.$$

To show that $G \otimes S <_r G \otimes T$, it suffices to show that τ_1 is an injective regular relational morphism. Since $\tau : S \to T$ is injective, we then clearly have that $(\varphi,s)\tau_1 \neq \phi$ for all $(\varphi,s) \in G \otimes S$. Let $(\varphi_1,s_1),(\varphi_2,s_2) \in G \otimes S$, $(\phi_1,t_1) \in (\varphi_1,s_1)\tau_1$ and $(\phi_2,t_2) \in (\varphi_2,s_2)\tau_1$. Then $t_1 \in s_1\tau$ and $t_2 \in s_2\tau$ so that $t_1t_2 \in (s_1s_2)\tau$. Since $(\varphi_1,s_1)(\varphi_2,s_2) = (\varphi_1\, {}^{s_1}\varphi_2, s_1s_2)$, $(\phi_1,t_1)(\phi_2,t_2) = (\phi_1\, {}^{t_1}\phi_2, t_1t_2)$ and

$$\begin{aligned} 1_T(\phi_1\, {}^{t_1}\phi_2) &= (1_T\phi_1)(t_1\phi_2) \\ &= (1_S\varphi_1)(s_1\varphi_2) \\ &= 1_S(\varphi_1\, {}^{s_1}\varphi_2), \end{aligned}$$

and if $x \in y\tau$, then $xt_1 \in (y\tau)(s_1\tau) \subseteq (ys_1)\tau$ and

$$\begin{aligned} x(\phi_1\, {}^{t_1}\phi_2) &= (x\phi_1)(xt_1\phi_2) \\ &= (y\varphi_1)(ys_1\varphi_2) \\ &= y(\varphi_1\, {}^{s_1}\varphi_2), \end{aligned}$$

it follows that $(\phi_1,t_1)(\phi_2,t_2) \in ((\varphi_1,s_1)(\varphi_2,s_2))\tau_1$. Hence $(\varphi_1,s_1)\tau_1(\varphi_2,s_2)\tau_1 \subseteq ((\varphi_1,s_1)(\varphi_2,s_2))\tau_1$, and therefore τ_1 is a relational morphism.

We now show that τ_1 is regular. Let $(\varphi,s)\in G\otimes S$ and $(\phi,t)\in(\varphi,s)\tau_1$. Then $t\in s\tau$, and there exist $s'\in V(s)$ and $t'\in V(t)$ such that $t'\in s'\tau$. Let $\varphi_1 : S^1\to G$ be defined by $y\varphi_1=(ys'\varphi)^{-1}$ for all $y\in S^1$, and $\phi_1 : T^1\to G$ be defined by $x\phi_1=(xt'\phi)^{-1}$ for all $x\in T^1$. Routine calculations now show that $(\varphi_1,s')\in V((\varphi,s))$ and $(\phi_1,t')\in V((\phi,t))$. Since

$$1_T\phi_1=(1_Tt'\phi)^{-1}=(1_Ss'\varphi)^{-1}=1_S\varphi_1,$$

and if $x\in y\tau$, then $xt'\in(ys')\tau$ and

$$x\phi_1=(xt'\phi)^{-1}=(ys'\varphi)^{-1}=y\varphi_1,$$

it follows that $(\phi_1,t')\in(\varphi_1,s')\tau_1$. Hence, $g(\tau_1)$ is a regular subsemigroup of $(G\otimes S)\times(G\otimes T)$, and so τ_1 is regular.

We finally show that τ_1 is injective. If $(\phi,t)\in(\varphi_1,s_1)\tau_1\cap(\varphi_2,s_2)\tau_1$, then $t\in s_1\tau\cap s_2\tau$ so that $s_1=s_2$, since τ is injective. For each $y\in S$, there exists $x\in T$ such that $x\in y\tau$. Thus $y\varphi_1=x\phi=y\varphi_2$ and $1_S\varphi_1=1_T\phi=1_S\varphi_2$, that is, $\varphi_1=\varphi_2$. Hence τ_1 is injective.

(b) $S\otimes G<_rT\otimes G$. For each $(\varphi,g)\in S\otimes G$, we let

$$(\varphi,g)\tau_2=\{(\phi,g)\in T\otimes G\mid x\phi\in(x\varphi)\tau \text{ for all } x\in\mathscr{G}\}.$$

Clearly, $(\varphi,g)\tau_2\neq\phi$ for all $(\varphi,g)\in S\otimes G$. Arguments similar to those in (a) prove that τ_2 is also an injective regular relational morphism from $S\otimes G$ to $T\otimes G$. Hence, $S\otimes G<_rT\otimes G$. □

Lemma 3.9. (a) *Let $G,G_1\in\mathscr{G}$ be such that $G<_rG_1$. Then $G\otimes R<_rG_1\otimes R$ for any $R\in\mathscr{RS}$.*

(b) *Let $S,T\in\mathscr{CS}$ be such that there is an embedding $\tau : S\to T$. Then* Reg$(S\otimes R)$ $<_r$Reg$(T\otimes R)$ *for any $R\in\mathscr{RS}$.*

Proof. (a) Let $R\in\mathscr{RS}$. Since $G<_rG_1$, then by Result 3.1 there is an injective regular relational morphism $\tau : G\to G_1$. For each $(\varphi,r)\in G\otimes R$, we let

$$(\varphi,r)\tau_1=\{(\phi,r)\in G_1\otimes R\mid x\phi\in(x\varphi)\tau \text{ for all } x\in R^1\}.$$

The same type of argument as in the proof of Lemma 3.8 yields that τ_1 is an injective regular relational morphism from $G\otimes R$ to $G_1\otimes R$, so that $G\otimes R<_rG_1\otimes R$.

(b) Let $R\in\mathscr{RS}$. To show that Reg$(S\otimes R)<_r$Reg$(T\otimes R)$, it suffices to show that there is an embedding $\tau_1 : S\otimes R\to T\otimes R$.

For each $(\varphi,r)\in S\otimes R$, we let $(\varphi,r)\tau_1=(\varphi\tau,r)$, where $\varphi\tau : R^1\to T$ is defined by $x(\varphi\tau)=(x\varphi)\tau$, for all $x\in R^1$. Clearly, τ_1 maps $S\otimes R$ into $T\otimes R$. If $(\varphi_1,r_1),(\varphi_2,r_2)\in S\otimes R$, then

$$\begin{aligned}(\varphi_1,r_1)\tau_1(\varphi_2,r_2)\tau_1&=(\varphi_1\tau,r_1)(\varphi_2\tau,r_2)\\&=(\varphi_1\tau\,{}^{r_1}(\varphi_2\tau),r_1r_2),\end{aligned}$$

and for any $x \in R^1$, we have

$$\begin{aligned} x(\varphi_1 \tau^{r_1}(\varphi_2 \tau)) &= (x\varphi_1)\tau(xr_1\varphi_2)\tau \\ &= ((x\varphi_1)(xr_1\varphi_2))\tau \\ &= x(\varphi_1{}^{r_1}\varphi_2)\tau, \end{aligned}$$

so that

$$\begin{aligned} (\varphi_1, r_1)\tau_1(\varphi_2, r_2)\tau_1 &= (\varphi_1\tau^{r_1}(\varphi_2\tau), r_1 r_2) \\ &= ((\varphi_1{}^{r_1}\varphi_2)\tau, r_1 r_2) \\ &= (\varphi_1{}^{r_1}\varphi_2, r_1 r_2)\tau_1 \\ &= ((\varphi_1, r_1)(\varphi_2, r_2))\tau_1, \end{aligned}$$

whence τ_1 is a homomorphism. If $(\varphi_1, r_1)\tau_1 = (\varphi_2, r_2)\tau_1$, then $r_1 = r_2$ and $\varphi_1\tau = \varphi_2\tau$, that is, $(x\varphi_1)\tau = (x\varphi_2)\tau$ for all $x \in R^1$. Since τ is one-to-one, it follows that $\varphi_1 = \varphi_2$. Hence τ_1 is one-to-one, and therefore τ_1 is an embedding. □

In order to describe the operator K_ℓ, the authors introduced a special type of wreath product structure in [25]. Let Y be a non-trivial semilattice with identity 1 and let T be an arbitrary regular semigroup. Then the wreath product $Y \otimes T$ of Y and T is not necessarily a regular semigroup. Let

$$(Y \otimes T)^* = \{(f, t) \in Y \otimes T \mid \text{for } x \in T,\ x \mathscr{R} xt \text{ implies } xf = 1\}.$$

Then by [25, Lemmas 6.1 and 6.2], $(Y \otimes T)^*$ is a regular subsemigroup of $Y \otimes T$ such that the projection of $(Y \otimes T)^*$ onto T induces a congruence over $\mathscr{L}\mathscr{Z}$. For any $\mathscr{U} \in \mathscr{L}_{er}(\mathscr{R}\mathscr{S})$, we let

$$\mathscr{S} \otimes^* \mathscr{U} = \langle (Y^1 \otimes T)^* \mid Y^1 \in \mathscr{S} \text{ and } T \in \mathscr{U} \rangle_{er}.$$

Then we have the following results.

Result 3.10 (Reilly and Zhang [25]). *Let $\mathscr{U} \in \mathscr{L}_{er}(\mathscr{R}\mathscr{S})$. Then*

$$\mathscr{S} \otimes^* \mathscr{U} = \begin{cases} \mathscr{U} & \text{if } \mathscr{U} \subseteq \mathscr{G}; \\ \mathscr{U}^{K_\ell} & \text{otherwise.} \end{cases}$$

Lemma 3.11. *Let $\mathscr{U} \in \mathscr{L}_{er}(\mathscr{E}\mathscr{S})$. Then*

$$\mathscr{S} \otimes^* \mathscr{U} = \{S \in \mathscr{R}\mathscr{S} \mid S <_r (Y^1 \otimes T)^* \text{ for some } Y \in \mathscr{S} \text{ and } T \in \mathscr{U}\}.$$

Proof. Let $\mathscr{W} = \{(Y^1 \otimes T)^* \mid Y \in \mathscr{S} \text{ and } T \in \mathscr{U}\}$. It is clear that for any indexed families $\{Y_i\}_{i\in I}$, $\{T_i\}_{i\in I}$ of regular semigroups such that for each $i \in I$, $Y_i \in \mathscr{S}$ is a monoid, there is a natural embedding from $\prod_{i\in I}(Y_i \otimes T_i)$ into $(\prod_{i\in I} Y_i) \otimes (\prod_{i\in I} T_i)$. It follows easily that

$$\prod_{i\in I}(Y_i \otimes^* T_i) <_r \left(\prod_{i\in I} Y_i\right) \otimes^* \left(\prod_{i\in I} T_i\right),$$

so that $\mathbf{P}\mathcal{W} \subseteq \mathbf{S}_r\mathcal{W}$. By Result 3.10 and [24, Lemma 7.4], $\mathcal{U} \subseteq \mathcal{ES}$ implies that $\mathcal{S} \otimes^* \mathcal{U} \subseteq \mathcal{U}^{K_l} \subseteq \mathcal{ES}$. Making use of [33, Lemma 4.8], we therefore obtain that

$$\mathcal{S} \otimes^* \mathcal{U} = \mathbf{HS}_r\mathbf{P}\mathcal{W} \subseteq \mathbf{HS}_r\mathbf{S}_r\mathcal{W} = \mathbf{HS}_r\mathcal{W} \subseteq \mathcal{S} \otimes^* \mathcal{U},$$

that is, $\mathcal{S} \otimes^* \mathcal{U} = \mathbf{HS}_r\mathcal{W}$, as required. □

Lemma 3.12. *Let $S, T \in \mathcal{RS}$ be such that $S <_r T$. Then*

$$(Y^1 \otimes S)^* <_r (Y^1 \otimes T)^*$$

for any $Y \in \mathcal{S}$.

Proof. Since $S <_r T$, then by Result 3.1, there exists an injective regular relational morphism $\tau : S \to T$. Let $Y \in \mathcal{S}$. Similar to the proof of Lemma 3.8(a), we let

$$(\varphi, s)\tau_1 = \left\{ (\phi, t) \in (Y^1 \otimes T)^* \;\middle|\; \begin{array}{c} t \in s\tau \text{ and } \phi \text{ satisfies} \\ \text{(i) } 1_T\phi = 1_S\phi; \\ \text{(ii) } x \in y\tau \text{ implies } x\phi = y\varphi \end{array} \right\}$$

for any $(\varphi, s) \in (Y^1 \otimes S)^*$.

We first show that $(\varphi, s)\tau_1 \neq \phi$. Let $t \in s\tau$. Then there exist $t' \in V(t)$ and $s' \in V(s)$ such that $t' \in s'\tau$. Define $\phi : T^1 \to Y^1$ by

$$x\phi = \begin{cases} 1_S\varphi & \text{if } x = 1_T, \\ y\varphi & \text{if } x \in y\tau \text{ for some } y \in S, \\ 1 & \text{otherwise.} \end{cases}$$

Since τ is injective, ϕ is well-defined, so that $(\phi, t) \in Y^1 \otimes T$. To show that $(\phi, t) \in (\varphi, s)\tau_1$, it suffices to show that $(\phi, t) \in (Y^1 \otimes T)^*$. So let $x \in T^1$ be such that $x \mathcal{R} xt$. If $x \in y\tau$ for some $y \in S$, then $xt \in (ys)\tau$, so that there exist $(xt)' \in V(xt)$ and $(ys)' \in V(ys)$ such that $(xt)' \in (ys)'\tau$. Also, $x \mathcal{R} xt$ implies that

$$x = (xt)(xt)'x \in y\tau \cap (ys(ys)'y)\tau,$$

it follows that $y = ys(ys)'y$. Thus $y \mathcal{R} ys$, and so $x\phi = y\varphi = 1$. Hence $(\phi, t) \in (Y^1 \otimes T)^*$, and therefore $(\varphi, s)\tau_1 \neq \phi$.

The rest of the proof proceeds exactly as in the proof of Lemma 3.8. □

We conclude this section with the following useful observations.

Proposition 3.13. (a) *For any $\mathcal{V} \in \mathcal{L}_{er}(\mathcal{RS})$, we have*

$$\mathcal{RB} \otimes V = (\mathcal{LZ} \otimes \mathcal{V}) \vee (\mathcal{RZ} \otimes \mathcal{V}).$$

(b) *For any $\mathcal{U} \in \mathcal{L}_{er}(\mathcal{R}e\mathcal{G})$ and any $\mathcal{V} \in \mathcal{L}_{er}(\mathcal{RS})$, we have*

$$\mathcal{U} \otimes \mathcal{V} = ((\mathcal{U} \cap \mathcal{RB}) \otimes \mathcal{V}) \vee ((\mathcal{U} \cap \mathcal{G}) \otimes \mathcal{V}).$$

In particular, $\mathcal{R}e\mathcal{G} \otimes \mathcal{V} = \mathcal{RB} \otimes \mathcal{V} \vee \mathcal{G} \otimes \mathcal{V}$.

(c) *For any* $\mathscr{V} \in \mathscr{L}_{ev}(\mathscr{G})$ *and any family* $\{\mathscr{U}_\alpha\}_{\alpha\in A}$ *of e-varieties such that* $\bigvee_{\alpha\in A} \mathscr{U}_\alpha \subseteq \mathscr{I}L$ *or* $\mathscr{E}\mathscr{S}$, *we have*

$$\left(\bigvee_{\alpha\in A} \mathscr{U}_\alpha\right) \otimes \mathscr{V} = \bigvee_{\alpha\in A} \mathscr{U}_\alpha \otimes \mathscr{V}.$$

Proof. It is well known that for any indexed family $\{S_i\}_{i\in I}$ of regular semigroups and any regular semigroup T, there is a natural embedding from $(\prod_{i\in I} S_i)\otimes T$ into $\prod_{i\in I}(S_i\otimes T)$, so that $\text{Reg}((\prod_{i\in I} S_i)\otimes T) <_r \prod_{i\in I}\text{Reg}(S_i\otimes T)$ whenever $\text{Reg}((\prod_{i\in I} S_i)\otimes T)$ and $\text{Reg}(S_i\otimes T)$ $(i\in I)$ are subsemigroups.

(a) Since any rectangular band is the direct product of a left zero semigroup and a right zero semigroup, it follows from the above remark that

$$\begin{aligned}
\mathscr{R}\mathscr{B}\otimes\mathscr{V} &= \langle \text{Reg}(S\otimes T) \mid S\in\mathscr{R}\mathscr{B}, T\in\mathscr{V}\rangle_{ev}\\
&= \langle \text{Reg}((L\times R)\otimes T) \mid L\in\mathscr{L}\mathscr{Z}, R\in\mathscr{R}\mathscr{Z}, T\in\mathscr{V}\rangle_{ev}\\
&\subseteq \langle \text{Reg}(L\otimes T)\times\text{Reg}(R\otimes T) \mid L\in\mathscr{L}\mathscr{Z}, R\in\mathscr{R}\mathscr{Z}, T\in\mathscr{V}\rangle_{ev}\\
&\subseteq (\mathscr{L}\mathscr{Z}\otimes\mathscr{V})\vee(\mathscr{R}\mathscr{Z}\otimes\mathscr{V}).
\end{aligned}$$

The converse is obvious.

(b) This follows similarly as in (a) by using the fact that any rectangular group is the direct product of a rectangular band and a group.

(c) Let $\mathscr{U}_\alpha\in\mathscr{L}_{ev}(\mathscr{R}\mathscr{S})$ $(\alpha\in A)$ and $\mathscr{V}\in\mathscr{L}_{ev}(\mathscr{G})$ be such that $\bigvee_{\alpha\in A}\mathscr{U}_\alpha\subseteq\mathscr{I}L$ or $\mathscr{E}\mathscr{S}$. Then by [33, Lemma 4.8], we have

$$\bigvee_{\alpha\in A}\mathscr{U}_\alpha = \left\{ S\in\mathscr{R}\mathscr{S} \;\middle|\; S<_r \prod_{\alpha\in A} S_\alpha \text{ for some } S_\alpha\in\mathscr{U}_\alpha(\alpha\in A)\right\}.$$

It follows by Lemma 3.8 and the above remarks that

$$\begin{aligned}
\left(\bigvee_{\alpha\in A}\mathscr{U}_\alpha\right)\otimes\mathscr{V} &= \left\langle S\otimes T \;\middle|\; S\in\bigvee_{\alpha\in A}\mathscr{U}_\alpha, T\in\mathscr{V}\right\rangle_{ev}\\
&\subseteq \left\langle \left(\prod_{\alpha\in A} S_\alpha\right)\otimes T \;\middle|\; S_\alpha\in\mathscr{U} \text{ for each } \alpha\in A \text{ and } T\in\mathscr{V}\right\rangle_{ev}\\
&\subseteq \bigvee_{\alpha\in A}\langle S_\alpha\otimes T \mid S_\alpha\in\mathscr{U}_\alpha, T\in\mathscr{V}\rangle_{ev}\\
&= \bigvee_{\alpha\in A}\mathscr{U}_\alpha\otimes\mathscr{V}.
\end{aligned}$$

The opposite inclusion is obvious. □

4. Some associativity results

Using the relationships between semigroups and complete transformations, Eilenberg [3] and Tilson [30] proved that the (standard) wreath product of (pseudo)-varieties of (finite) semigroups satisfies the associative law. For a slightly different wreath product, Koselev [15] also announced that the associative law holds. However, Peter Jones has drawn our attention to the fact that there appear to be some gaps in his proofs. In this section we modify Koselev's techniques and establish some associativity results on the products of e-varieties introduced in Section 3.

Lemma 4.1. *Let* $S, T, R \in \mathscr{RS}$.

(a) *There is an embedding* $\tau_1 : (S \otimes T) \otimes R \to S \otimes (T^1 \otimes R^1)$.

(b) *If* $S \in \mathscr{G}$, *then there is a homomorphism* $\tau_2 : (S \otimes T) \otimes R^1 \to S \otimes (T \otimes R^1)$ *such that* $\overline{\tau_2}$ *is over* $\mathscr{LZ}$.

Proof. (a) For each $(g,r) \in (S \otimes T) \otimes R$ with $xg = (\beta_x, t_x)$ for all $x \in R^1$, we let $(g,r)\tau_1 = (\varphi, (\psi, r))$, where $\psi : R^1 \to T^1$ is defined by $x\psi = t_x$ for all $x \in R^1$ and $\varphi : T^1 \otimes R^1 \to S$ is defined by $(p,x)\varphi = (1p)\beta_x$ for all $(p,x) \in T^1 \otimes R^1$ (note that $T^1 \otimes R^1$ is a monoid). Clearly, τ_1 is well-defined.

We first show that τ_1 is a homomorphism. So let $(g,r),(g_1,r_1) \in (S \otimes T) \otimes R$ with $xg = (\beta_x, t_x)$ and $xg_1 = (\gamma_x, s_x)$ for all $x \in R^1$, and let $(g,r)\tau_1 = (\varphi, (\psi, r)), (g_1, r_1)\tau_1 = (\varphi_1, (\psi_1, r_1))$. Then

$$(g,r)(g_1,r_1) = (g^r g_1, rr_1)$$

where, for all $x \in R^1$,

$$\begin{aligned} x(g^r g_1) &= (xg)(xrg_1) \\ &= (\beta_x, t_x)(\gamma_{xr}, s_{xr}) \\ &= (\beta_x\, {}^{t_x}\gamma_{xr}, t_x s_{xr}). \end{aligned}$$

If we let $((g,r)(g_1,r_1))\tau_1 = (\varphi_2, (\psi_2, rr_1))$, then we have

$$x\psi_2 = t_x s_{xr} \quad \text{for all } x \in R^1$$

and

$$\begin{aligned} (p,x)\varphi_2 &= (1p)(\beta_x\, {}^{t_x}\gamma_{xr}) \\ &= ((1p)\beta_x)((1p)t_x\gamma_{xr}) \quad \text{for all } (p,x) \in T^1 \otimes R^1. \end{aligned} \tag{1}$$

Now

$$\begin{aligned} (g,r)\tau_1(g_1,r_1)\tau_1 &= (\varphi, (\psi, r))(\varphi_1, (\psi_1, r_1)) \\ &= (\varphi^{(\psi,r)}\varphi_1, (\psi^r\psi_1, rr_1)), \end{aligned} \tag{2}$$

where, for any $x \in R^1$, we have

$$x(\psi^r\psi_1) = (x\psi)(xr\psi_1) = t_x s_{xr} = x\psi_2,$$

that is, $\psi^r\psi_1 = \psi_2$. For any $(p,x) \in T^1 \otimes R^1$, we have

$$\begin{aligned}(p,x)(\varphi^{(\psi,r)}\varphi_1) &= (p,x)\varphi((p,x)(\psi,r))\varphi_1\\ &= (1p)\beta_x(p^x\psi,xr)\varphi_1\\ &= (1p)\beta_x(1p^x\psi)\gamma_{xr}\\ &= (1p)\beta_x((1p)t_x)\gamma_{xr}\\ &= (p,x)\varphi_2,\end{aligned}$$

that is, $\varphi^{(\psi,r)}\varphi_1 = \varphi_2$. Thus by (1) and (2), $(g,r)\tau_1(g_1,r_1)\tau_1 = ((g,r)(g_1,r_1))\tau_1$ and hence τ_1 is a homomorphism.

We now show that τ_1 is one-to-one. If $(g,r)\tau_1 = (g_1,r_1)\tau_1$, that is, $(\varphi,(\psi,r)) = (\varphi_1,(\psi_1,r_1))$. It follows that $r = r_1$, $\psi = \psi_1$ and $\varphi = \varphi_1$. For any $x \in R^1$, we then have $t_x = x\psi = x\psi_1 = s_x$ and for any mapping $p : R^1 \to T^1$, we have

$$(1p)\beta_x = (p,x)\varphi = (p,x)\varphi_1 = (1p)\gamma_x.$$

Since $T^1 = \cup_p\ 1p$, it follows that $\beta_x = \gamma_x$ for all $x \in R^1$. Thus $xg = (\beta_x,t_x) = (\gamma_x,s_x) = xg_1$ for all $x \in R^1$, that is, $g = g_1$. Hence $(g,r) = (g_1,r_1)$, and therefore τ_1 is an embedding.

(b) Let $S \in \mathcal{G}$. If T is a monoid, the assertion follows from (a). So we assume that T is not a monoid, which implies that $T \otimes R^1$ is also not a monoid. As in (a), for each $(g,r) \in (S \otimes T) \otimes R^1$, let $(g,r)\tau_2 = (\varphi,(\psi,r))$, where $\psi : R^1 \to T$ is defined by $x\psi = t_x$ for all $x \in R^1$ and

$$\psi : (T \otimes R^1)^1 \to S$$

is defined by $(p,x)\varphi = (1p)\beta_x$ for all $(p,x) \in T \otimes R^1$, and $1\varphi = 1\beta_1$. Clearly, τ_2 is a mapping of $(S \otimes T) \otimes R^1$ into $S \otimes (T \otimes R^1)$. As in (a), it is a straightforward verification that τ_2 is a homomorphism.

We now show that $\overline{\tau_2}$ is over $\mathcal{LZ}$. So let $(g,r),(g_1,r_1) \in (S \otimes T) \otimes R^1$ with $xg = (\beta_x,t_x)$ and $xg_1 = (\gamma_x,s_x)$ for all $x \in R^1$, be such that $(g,r)\tau_2 = (g_1,r_1)\tau_2 = (\varphi,(\psi,r)) = (\varphi_1,(\psi_1,r_1))$ is an idempotent of $S \otimes (T \otimes R^1)$. Then $\varphi = \varphi_1$, $\psi = \psi_1$ and $r = r_1$. Since $(\varphi,(\psi,r))(\varphi,(\psi,r)) = (\varphi^{(\psi,r)}\varphi,(\psi^r\psi,r^2)) = (\varphi,(\psi,r))$, we have $\varphi = \varphi^{(\psi,r)}\varphi$, $\psi = \psi^r\psi$ and $r \in E(R^1)$. Thus for any $(p,x) \in T \otimes R^1$, we have

$$\begin{aligned}(1p)\beta_x &= (p,x)\varphi\\ &= (p,x)(\varphi^{(\psi,r)}\varphi)\\ &= (p,x)\varphi((p,x)(\psi,r))\varphi\\ &= (p,x)\varphi(p^x\psi,xr)\varphi_1\end{aligned}$$

$$= (1p)\beta_x((1p)(x\psi))\gamma_{xr}$$

$$= (1p)\beta_x((1p)t_x)\gamma_{xr},$$

so that $((1p)t_x)\gamma_{xr} = 1$, the identity of S (S is a group). Let $p : R^1 \to T$ be such that $1p = t_x t'_x$ for some $t'_x \in V(t_x)$ in T. It follows that $t_x\gamma_{xr} = 1$ for all $x \in R^1$.

Now we have $(g,r)(g_1,r) = (g^r g_1, r)$ and for any $x \in R^1$,

$$\begin{aligned} x(g^r g_1) &= (xg)(xrg_1) \\ &= (\beta_x, t_x)(\gamma_{xr}, s_{xr}) \\ &= (\beta_x{}^{t_x}\gamma_{xr}, t_x s_{xr}). \end{aligned} \tag{3}$$

Since $\psi = \psi_1$ and $\psi = \psi^r\psi$, then for any $x \in R^1$, we have

$$t_x s_{xr} = (x\psi)(xr\psi_1) = x(\psi^r\psi) = x\psi = t_x. \tag{4}$$

Also, $\varphi = \varphi_1$ and $\varphi = \varphi^{(\psi,r)}\varphi$ imply that

$$\begin{aligned} (p,x)\varphi &= (1p)\beta_x \\ &= (p,x)(\varphi^{(\psi,r)}\varphi) \\ &= (p,x)\varphi((p,x)(\psi,r))\varphi_1 \\ &= (1p)\beta_x(p^x\psi, xr)\varphi_1 \\ &= (1p)\beta_x((1p)t_x)\gamma_{xr} \quad \text{for any } (p,x) \in T \otimes R^1. \end{aligned}$$

It follows that for any $y \in T$,

$$y\beta_x = (y\beta_x)(yt_x)\gamma_{xr} = y(\beta_x{}^{t_x}\gamma_{xr})$$

and

$$\begin{aligned} 1\beta_x &= (1\beta_x)(t_x\gamma_{xr}) \quad \text{since } t_x\gamma_{xr} = 1 \\ &= 1(\beta_x{}^{t_x}\gamma_{xr}), \end{aligned}$$

that is, $\beta_x{}^{t_x}\gamma_{xr} = \beta_x$ for all $x \in R^1$. It follows from (3) and (4) that $(g,r)(g_1,r) = (g,r)$, whence $(\varphi,(\psi,r))\tau^{-1}$ is a left zero semigroup. Hence $\overline{\tau_2}$ is over $\mathscr{L}\mathscr{Z}$, as required. □

Corollary 4.2. *Let $G \in \mathscr{G}$, $T \in \mathscr{CS}$ and $R \in \mathscr{RS}$ be such that R is a monoid. Then there is a homomorphism*

$$\tau : \mathrm{Reg}((G \otimes T) \otimes R) \to G \otimes \mathrm{Reg}(T \otimes R)$$

such that $\overline{\tau}$ is over $\mathscr{L}\mathscr{Z}$.

Proof. As in the proof of Lemma 4.1(b), for each $(g,r) \in \mathrm{Reg}((G \otimes T) \otimes R)$, let $(g,r)\tau = (\varphi,(\psi,r))$, where $\psi : R \to T$ is defined by $x\psi = t_x$ for all $x \in R$ and $\varphi : \mathrm{Reg}(T \otimes R) \to G$

is defined by $(p,x)\varphi=(1p)\beta_x$ for all $(p,x)\in \operatorname{Reg}(T\otimes R)$, and $1\varphi=1\beta_1$. Since $(g,r)\in \operatorname{Reg}((G\otimes T)\otimes R)$, it follows easily that $(\psi,r)\in \operatorname{Reg}(T\otimes R)$, so that $(g,r)\tau=(\varphi,(\psi,r))\in G\otimes \operatorname{Reg}(T\otimes R)$. Since for any fixed $t\in T$, if we let $p_t:R\to T$ by $yp_t=t$ for all $y\in R$, then we have $(p_t,x)\in \operatorname{Reg}(T\otimes R)$ for all $x\in R$, it follows from the proof of Lemma 4.1(b) that τ is a homomorphism of $\operatorname{Reg}((G\otimes T)\otimes R)$ into $G\otimes \operatorname{Reg}(T\otimes R)$ such that $\bar{\tau}$ is over $\mathscr{L}\mathscr{Z}$. □

Lemma 4.3. *Let $S,R,T\in\mathscr{R}\mathscr{S}$ be such that R is a monoid. Then*

$$S\otimes \operatorname{Reg}(T\otimes R)<_r \operatorname{Reg}((S\otimes T^R)\otimes R)$$

if (a) $S\in\mathscr{G}$, $T\in\mathscr{C}\mathscr{S}$; *or* (b) $S,R\in\mathscr{G}$; *or* (c) $T,R\in\mathscr{G}$.

Proof. Note that, by Result 3.2, in all three cases both $S\otimes \operatorname{Reg}(T\otimes R)$ and $\operatorname{Reg}((S\otimes T^R)\otimes R)$ are regular. Also, T^R is a monoid if and only if T is a monoid.

We define $\tau:S\otimes \operatorname{Reg}(T\otimes R)\to \operatorname{Reg}((S\otimes T^R)\otimes R)$ by: for each $(\varphi,(\psi,r))\in S\otimes \operatorname{Reg}(T\otimes R)$ so that

$$\varphi:\operatorname{Reg}(T\otimes R)^1\to S,\quad \psi:R\to T,\ r\in R$$

we set

$$(\varphi,(\psi,r))\tau=\left\{(g,r)\in \operatorname{Reg}((S\otimes T^R)\otimes R)\ \middle|\ \begin{array}{l} xg=(\beta_x,h_x)\text{ for all }x\in R,\\ \text{where}\\ \text{(i) } h_x={}^x\psi:R\to T;\\ \text{(ii) } \beta_x:(T^R)^1\to S\quad\text{with}\\ p\beta_x=(p,x)\varphi,\text{ if }p\in T^R\text{ and}\\ (p,x)\in \operatorname{Reg}(T\otimes R);\\ \text{(iii) } 1\beta_1=1\varphi,\text{ if }T\text{ is not a monoid}\end{array}\right\}.$$

Note that τ is not, in general, a function since $1\beta_x$ is unspecified except for $x=1$, the identity of R. In cases (b) and (c), it follows from Result 3.2 that $S\otimes \operatorname{Reg}(T\otimes R)=S\otimes(T\otimes R)$ and $\operatorname{Reg}((S\otimes T^R)\otimes R)=(S\otimes T^R)\otimes R$, and so clearly $(\varphi,(\psi,r))\tau\neq\emptyset$. In case (a), we let $(g,r)\in(S\otimes T^R)\otimes R$ with $xg=(\beta_x,h_x)$ for all $x\in R$, where $h_x={}^x\psi$ and $\beta_x:(T^R)^1\to S$ is given by

$$p\beta_x=\begin{cases}(p,x)\varphi & \text{if } p\in T^R \text{ and } (p,x)\in \operatorname{Reg}(T\otimes R);\\ 1\varphi & \text{otherwise,}\end{cases}$$

where $p\in(T^R)^1$. To show that $(g,r)\in(\varphi,(\psi,r))\tau$, it suffices to show that $(g,r)\in \operatorname{Reg}((S\otimes T^R)\otimes R)$. Since $(\varphi,(\psi,r))\in S\otimes \operatorname{Reg}(T\otimes R)$, it follows that $(\psi,r)\in \operatorname{Reg}(T\otimes R)$, and hence there exists $(\psi_1,r')\in \operatorname{Reg}(T\otimes R)$ such that $(\psi_1,r')\in V((\psi,r))$, so that $\psi=\psi^r\psi_1{}^{rr'}\psi$ and $r'\in V(r)$ in R. We now let $(g_1,r')\in(S\otimes T^R)\otimes R$ with $xg_1=(\gamma_x,f_x)$ for all $x\in R$, where $f_x={}^x\psi_1$ and $\gamma_x:(T^R)^1\to S$ is given by $p\gamma_x=(pf_x\beta_{xr'})^{-1}$ in S. It is straightforward to verify that $(g_1,r')\in V((g,r))$ in $(S\otimes T^R)\otimes R$, whence

$(g,r) \in \mathrm{Reg}((S \otimes T^R) \otimes R)$ and $(g,r) \in ((\varphi,\psi),r)\tau$. Therefore $(\varphi,(\psi,r))\tau \neq \emptyset$, for all $(\varphi,(\psi,r)) \in S \otimes \mathrm{Reg}(T \otimes R)$.

We now show that τ is a relational morphism. So let $(\varphi,(\psi,r)),(\varphi_1,(\psi_1,r_1)) \in S \otimes \mathrm{Reg}(T \otimes R)$, $(g,r) \in (\varphi,(\psi,r))\tau$ and $(g_1,r_1) \in (\varphi_1,(\psi_1,r_1))$. For any $x \in R$, we let $xg = (\beta_x, h_x)$ and $xg_1 = (\gamma_x, f_x)$. Thus $(g,r)(g_1,r_1) = (g^r g_1, rr_1)$ and

$$\begin{aligned} x(g^r g_1) &= (xg)(xrg_1) \\ &= (\beta_x, h_x)(\gamma_{xr}, f_{xr}) \\ &= (\beta_x\, {}^{h_x}\gamma_{xr}, h_x f_{xr}) \quad \text{for any } x \in R. \end{aligned}$$

We want to show that $(g^r g_1, rr_1) \in ((\varphi,(\psi,r))(\varphi_1,(\psi_1,r_1)))\tau = (\varphi^{(\psi,r)}\varphi_1, (\psi^r\psi_1, rr_1))\tau$. Since $h_x f_{xr} = ({}^x\psi)({}^{xr}\psi_1) = {}^x(\psi^r\psi_1)$, it follows that (i) holds. For any $p \in T^R$ with $(p,x) \in \mathrm{Reg}(T \otimes R)$, we have

$$\begin{aligned} p(\beta_x\, {}^{h_x}\gamma_{xr}) &= (p\beta_x)(ph_x)\gamma_{xr} \\ &= (p\beta_x)(p\,{}^x\psi)\gamma_{xr} \\ &= (p\beta_x)(p\,{}^x\psi, xr)\varphi_1 \\ &= (p,x)\varphi((p,x)(\psi,r))\varphi_1 \quad \text{since } (\psi,r) \in \mathrm{Reg}(T \otimes R) \\ &= (p,x)(\varphi^{(\psi,r)}\varphi_1), \end{aligned}$$

so that (ii) holds. If T is not a monoid, then

$$\begin{aligned} 1(\beta_1\, {}^{h_1}\gamma_r) &= (1\beta_1)(h_1\gamma_r) \\ &= (1\beta_1)(\psi\gamma_r) \\ &= (1\varphi)(\psi,r)\varphi_1 \\ &= 1(\varphi^{(\psi,r)}\varphi_1), \end{aligned}$$

so that (iii) holds. Hence $(g,r)(g_1,r_1) = (g^r g_1, rr_1) \in (\varphi^{(\psi,r)}\varphi_1, (\psi^r\psi_1, rr_1))\tau$, and therefore τ is a relational morphism.

We now show that τ is injective. If $(g,r) \in (\varphi,(\psi,r))\tau \cap (\varphi_1,(\psi_1,r_1))\tau$, then we have $r = r_1$, $h_x = {}^x\psi = f_x = {}^x\psi_1$ for all $x \in R$, so that $\psi = \psi_1$, by letting $x = 1$, the identity of R. Also, for any $(p,x) \in \mathrm{Reg}(T \otimes R)$,

$$(p,x)\varphi = p\beta_x = p\gamma_x = (p,x)\varphi_1$$

and if T is not a monoid,

$$1\varphi = 1\beta_1 = 1\gamma_1 = 1\varphi_1 \quad \text{so that } \varphi = \varphi_1.$$

Thus $(\varphi,(\psi,r)) = (\varphi_1,(\psi_1,r_1))$, and hence τ is injective.

To show that τ is regular, we consider two cases:

(a) and (b): Note that S is a group. Let $(g,r) \in (\varphi,(\psi,r))\tau$. Since $(\psi,r) \in \mathrm{Reg}(T \otimes R)$, there exists $(\psi_1,r') \in \mathrm{Reg}(T \otimes R)$ such that $(\psi_1,r') \in V((\psi,r))$. It follows that $\psi = \psi^r\psi_1\,{}^{rr'}\psi$ and $r' \in V(r)$ in R. Define $\varphi_1 : \mathrm{Reg}(T \otimes R)^1 \to S$ by $p\varphi_1 = (p(\psi_1,r')\varphi)^{-1}$

in S, for all $p\in \mathrm{Reg}(T\otimes R)^1$. We claim that $(\varphi_1,(\psi_1,r'))\in V((\varphi,(\psi,r)))$. First we have

$$\begin{aligned}(\varphi,(\psi,r))(\varphi_1,(\psi_1,r'))(\varphi,(\psi,r)) &= (\varphi^{(\psi,r)}\varphi_1,(\psi^r\psi_1,rr'))(\varphi,(\psi,r))\\ &= (\varphi^{(\psi,r)}\varphi_1{}^{(\psi^r\psi_1,rr')}\varphi,(\psi^r\psi_1{}^{rr'}\psi,r))\\ &= (\varphi^{(\psi,r)}\varphi_1{}^{(\psi^r\psi_1,rr')}\varphi,(\psi,r)).\end{aligned}$$

For any $p\in \mathrm{Reg}(T\otimes R)^1$, we have

$$\begin{aligned}p(\varphi^{(\psi,r)}\varphi_1{}^{(\psi^r\psi_1,rr')}\varphi) &= (p\varphi)(p(\psi,r))\varphi_1(p(\psi^r\psi_1,rr'))\varphi\\ &= (p\varphi)(p(\psi,r)(\psi_1,r')\varphi)^{-1}(p(\psi^r\psi_1,rr'))\varphi\\ &= (p\varphi)(p(\psi^r\psi_1,rr')\varphi)^{-1}(p(\psi^r\psi_1,rr'))\varphi\\ &= p\varphi \quad \text{since } S \text{ is a group}\end{aligned}$$

it follows that $\varphi=\varphi^{(\psi,r)}\varphi_1{}^{(\psi^r\psi_1,rr')}\varphi$. Hence $(\varphi,(\psi,r))=(\varphi,(\psi,r))(\varphi_1,(\psi_1,r'))(\varphi,(\psi,r))$. Similarly, $(\varphi_1,(\psi_1,r'))=(\varphi_1,(\psi_1,r'))(\varphi,(\psi,r))(\varphi_1,(\psi_1,r'))$. Therefore $(\varphi_1,(\psi_1,r'))\in V((\varphi,(\psi,r))$. We now let $(g_1,r')\in(S\otimes T^R)\otimes R$ with $xg_1=(\gamma_x,f_x)$ for all $x\in R$, where $f_x={}^x\psi_1$ and $\gamma_x:(T^R)^1\to S$ is given by $p\gamma_x=(pf_x\beta_{xrr'})^{-1}$ in S, for all $p\in(T^R)^1$. We claim that $(g_1,r')\in V((g,r))$. First, we have

$$(g,r)(g_1,r')(g,r)=(g^rg_1{}^{rr'}g,r).$$

For any $x\in R$,

$$\begin{aligned}x(g^rg_1{}^{rr'}g) &= (xg)(xrg_1)(xrr'g)\\ &= (\beta_x,h_x)(\gamma_{xr},f_{xr})(\beta_{xrr'},h_{xrr'})\\ &= (\beta_x{}^{h_x}\gamma_{xr}{}^{h_xf_{xr}}\beta_{xrr'},h_xf_{xr}h_{xrr'}),\end{aligned} \tag{5}$$

while

$$h_xf_{xr}h_{xrr'}=({}^x\psi)({}^{xr}\psi_1)({}^{xrr'}\psi)=x(\psi^r\psi_1{}^{rr'}\psi)=x\psi=h_x,$$

and for any $p\in(T^R)^1$, we have

$$\begin{aligned}p(\beta_x{}^{h_x}\gamma_{xr}{}^{h_xf_{xr}}\beta_{xrr'}) &= (p\beta_x)(ph_x\gamma_{xr})(ph_xf_{xr}\beta_{xrr'})\\ &= (p\beta_x)(ph_xf_{xr}\beta_{xrr'})^{-1}(\beta h_xf_{xr}\beta_{xrr'})\\ &= p\beta_x \quad \text{since } S \text{ is a group,}\end{aligned}$$

whence $\beta_x=\beta_x{}^{h_x}\gamma_{xr}{}^{h_xf_{xr}}\beta_{xrr'}$. It follows from (5) that $(g,r)=(g,r)(g_1,r')(g,r)$. Similarly, $(g_1,r')(g,r)(g_1,r')=(g_1,r')$. Thus $(g_1,r')\in V((g,r))$. If $p\in T^R$ and $(p,x)\in$

$\text{Reg}(T \otimes R)$, we then have

$$\begin{aligned} p\gamma_x &= (pf_x\beta_{xr'})^{-1} \\ &= ((p^x\psi_1, xr')\varphi)^{-1} \quad \text{since } (p^x\psi_1, xr') \in \text{Reg}(T \otimes R) \\ &= ((p,x)(\psi_1, r')\varphi)^{-1} \\ &= (p,x)\varphi_1, \end{aligned}$$

so that (ii) holds. If T is not a monoid, then

$$\begin{aligned} 1\gamma_1 &= (f_1\beta_{r'})^{-1} \\ &= (\psi_1\beta_{r'})^{-1} \\ &= ((\psi_1, r')\varphi)^{-1} \quad \text{since } (\psi_1, r') \in \text{Reg}(T \otimes R) \\ &= 1\varphi_1, \end{aligned}$$

so that (iii) holds. Hence $(g_1, r') \in (\varphi_1, (\psi_1, r'))\tau$, and therefore τ is regular.

(c): Note that both T and R are groups. Let $(g, r) \in (\varphi, (\psi, r))\tau$. We define $\psi_1 : R \to T$ by $y\psi_1 = (yr^{-1}\psi)^{-1}$ in T, and let $\varphi_1 : T \otimes R \to S$ by

$$(p,x)\varphi_1 = ((p,x)(\psi_1, r^{-1})\varphi)'$$

for some $((p,x)(\psi_1, r^{-1})\varphi)' \in V((p,x)(\psi_1, r^{-1})\varphi)$ in S. Since $(\psi, r)(\psi_1, r^{-1}) = (\psi^r\psi_1, 1)$ and for any $y \in R$,

$$\begin{aligned} y(\psi^r\psi_1) &= (y\psi)(yr\psi_1) \\ &= (y\psi)(yrr^{-1}\psi)^{-1} \\ &= 1, \quad \text{the identity of } T, \end{aligned}$$

it follows that $(\psi, r)(\psi_1, r^{-1})$ is the identity of $T \otimes R$. Thus, by the same arguments as above, $(\varphi_1, (\psi_1, r^{-1})) \in V((\varphi, (\psi, r)))$. For each $x \in R$, we let $f_x = {}^x\psi_1$ and let $\gamma_x : T^G \to S$ be defined by $p\gamma_x = (p,x)\varphi_1$. Thus $(g_1, r^{-1}) \in (S \otimes T^R) \otimes R$ and $(g_1, r^{-1}) \in (\varphi_1, (\psi_1, r^{-1}))\tau$, where $xg_1 = (\gamma_x, f_x)$ for all $x \in R$. For any $y \in R$, we have

$$\begin{aligned} y(h_x f_{xr}) &= (yh_x)(yf_{xr}) \\ &= (yx\psi)(yxr\psi_1) \\ &= (yx\psi)(yxrr^{-1}\psi)^{-1} \\ &= 1, \quad \text{the identity of } T, \end{aligned}$$

it follows that $h_x f_{xr}$ is the identity of T^R for any $x \in R$. Again by the same arguments as in (a) and (b), we have $(g_1, r^{-1}) \in V((g, r))$. Hence τ is regular.

In all cases, we have shown that τ is regular. Therefore by Result 3.1, $S \otimes \text{Reg}(T \otimes R) <_r \text{Reg}((S \otimes T^R) \otimes R)$. $\square$

We are now ready to prove the following result.

Theorem 4.4. *Let $\mathscr{U}, \mathscr{V}, \mathscr{W} \in \mathscr{L}_{ev}(\mathscr{RS})$. Then the associative law*

$$\mathscr{U} \otimes (\mathscr{V} \otimes \mathscr{W}) = (\mathscr{U} \otimes \mathscr{V}) \otimes \mathscr{W}$$

holds for the following cases:

(a) $\mathscr{U} \vee \mathscr{W} \subseteq \mathscr{G}$ *and* $\mathscr{V} \subseteq \mathscr{ES}$;

(b) $\mathscr{V} \vee \mathscr{W} \subseteq \mathscr{G}$ *and* $\mathscr{U} \subseteq \mathscr{ES}$ *or* $\mathscr{I}L$.

Proof. (a) If $\mathscr{V} \in \mathscr{L}_{ev}(\mathscr{G})$, then the equality follows from [18, Theorem 21.51]. So we assume that $\mathscr{V} \notin \mathscr{L}_{ev}(\mathscr{G})$. Let $G \in \mathscr{U}$ and $R \in \mathscr{V} \otimes \mathscr{W}$. By Result 3.4, $\mathscr{V} \otimes \mathscr{W} \subseteq \mathscr{ES}$. Then by Result 3.3(b) and Result 3.2(a), $R <_r T \otimes G_1$ for some $T \in \mathscr{V}$ and $G_1 \in \mathscr{W}$. It follows from Lemma 3.8 that

$$G \otimes R <_r G \otimes (T \otimes G_1).$$

Thus by Lemma 4.3,

$$G \otimes R <_r G \otimes (T \otimes R) <_r (G \otimes T^R) \otimes R,$$

while the latter is in $(\mathscr{U} \otimes \mathscr{V}) \otimes \mathscr{W}$, it follows that $G \otimes R \in (\mathscr{U} \otimes \mathscr{V}) \otimes \mathscr{W}$. Hence $\mathscr{U} \otimes (\mathscr{V} \otimes \mathscr{W}) \subseteq (\mathscr{U} \otimes \mathscr{V}) \otimes \mathscr{W}$.

For the opposite inclusion, we let $R \in \mathscr{U} \otimes \mathscr{V}$ and $G \in \mathscr{W}$. We also assume that $\mathscr{U} \neq \mathscr{T}$, otherwise, it is trivial. By Results 3.3(b) and 3.5, $R <_r G_1 \otimes T$ for some $G_1 \in \mathscr{U}$ and $T \in \mathscr{V}$. It follows from Lemma 3.8 that $R \otimes G <_r (G_1 \otimes T) \otimes G$. There are two cases:

(i) If T is a monoid, then by Lemma 4.1(a),

$$R \otimes G <_r (G_1 \otimes T) \otimes G <_r G_1 \otimes (T \otimes G),$$

the latter is in $\mathscr{U} \otimes (\mathscr{V} \otimes \mathscr{W})$, it follows that $R \otimes G \in \mathscr{U} \otimes (\mathscr{V} \otimes \mathscr{W})$.

(ii) If T is not a monoid, then by Lemma 4.1(b), there is a homomorhism $\tau : (G_1 \otimes T) \otimes G \to G_1 \otimes (T \otimes G)$ such that $\bar{\tau}$ is over $\mathscr{LZ}$. Since $G_1 \otimes (T \otimes G) \in \mathscr{U} \otimes (\mathscr{V} \otimes \mathscr{W})$, it follows that $(G_1 \otimes T) \otimes G \in \mathscr{LZ} \circ (\mathscr{U} \otimes (\mathscr{V} \otimes \mathscr{W}))$, and therefore so does $R \otimes G$. Since $\mathscr{U} \in \mathscr{L}_{ev}(\mathscr{G}) \backslash \{\mathscr{T}\}$, $\mathscr{V} \notin \mathscr{L}_{ev}(\mathscr{G})$ and by Result 3.6, we have

$$\begin{aligned}
\mathscr{LZ} \circ (\mathscr{U} \otimes (\mathscr{V} \otimes \mathscr{W})) &= \mathscr{LZ} \circ \langle (\mathscr{LZ} \vee \mathscr{U}) \circ (\mathscr{V} \otimes \mathscr{W}) \rangle_{ev} \\
&\subseteq \langle (\mathscr{LZ} \circ (\mathscr{LZ} \vee \mathscr{U})) \circ (\mathscr{V} \otimes \mathscr{W}) \rangle_{ev} \quad \text{by Result 2.4} \\
&= \langle (\mathscr{LZ} \vee \mathscr{U}) \circ (\mathscr{V} \otimes \mathscr{W}) \rangle_{ev} \\
&= \mathscr{U} \otimes (\mathscr{V} \otimes \mathscr{W}),
\end{aligned}$$

it follows that $R \otimes G \in \mathscr{U} \otimes (\mathscr{V} \otimes \mathscr{W})$. Hence $(\mathscr{U} \otimes \mathscr{V}) \otimes \mathscr{W} \subseteq \mathscr{U} \otimes (\mathscr{V} \otimes \mathscr{W})$, and therefore $\mathscr{U} \otimes (\mathscr{V} \otimes \mathscr{W}) = (\mathscr{U} \otimes \mathscr{V}) \otimes \mathscr{W}$.

(b) This follows similarly by using Lemmas 4.1 and 4.3. □

Let $\mathscr{M}$ be the class of all regular monoids. For any $\mathscr{U} \in \mathscr{L}_{ev}(\mathscr{RS})$, we say that $\mathscr{U}$ is *monoidal* if $\mathscr{U} = \langle \mathscr{U} \cap \mathscr{M} \rangle_{ev}$. It follows from [26, Corollary 4.5] that $\mathscr{U}$ is monoidal if and only if $\mathscr{U}$ satisfies the condition that $S \in \mathscr{U}$ implies $S^1 \in \mathscr{U}$.

Theorem 4.5. *Let $\mathscr{W} \in \mathscr{L}_{er}(\mathscr{RS})$ be monoidal. Then*

$$\mathscr{U} \otimes (\mathscr{V} \otimes \mathscr{W}) = (\mathscr{U} \otimes \mathscr{V}) \otimes \mathscr{W}$$

for any $\mathscr{U}, \mathscr{V} \in \mathscr{L}_{er}(\mathscr{G})$.

Proof. Let $\mathscr{U}, \mathscr{V} \in \mathscr{L}_{er}(\mathscr{G})$. If $\mathscr{U} = \mathscr{T}$ or $\mathscr{V} = \mathscr{T}$, then the equality clearly holds. So we assume that $\mathscr{U} \neq \mathscr{T} \neq \mathscr{V}$. We also assume that $\mathscr{W} \notin \mathscr{L}_{er}(\mathscr{G})$, otherwise, the equality follows from Theorem 4.4. By Result 3.6, we have

$$\begin{aligned} \mathscr{U} \otimes (\mathscr{V} \otimes \mathscr{W}) &= \langle (\mathscr{LZ} \vee \mathscr{U}) \circ \langle (\mathscr{LZ} \vee \mathscr{V}) \circ \mathscr{W} \rangle_{er} \rangle_{er} \\ &\subseteq \langle \langle (\mathscr{LZ} \vee \mathscr{U}) \circ (\mathscr{LZ} \vee \mathscr{V}) \rangle_{er} \circ \mathscr{W} \rangle_{er} \quad \text{by Result 2.4} \\ &= \langle (\mathscr{LZ} \vee (\mathscr{U} \circ \mathscr{V})) \circ \mathscr{W} \rangle_{er} \\ &= (\mathscr{U} \circ \mathscr{V}) \otimes W \quad \text{by Result 3.6} \\ &= (\mathscr{U} \otimes \mathscr{V}) \otimes \mathscr{W} \quad \text{by [18, Section 2.2]}. \end{aligned}$$

For the opposite inclusion, we let $S \in \mathscr{U} \otimes \mathscr{V}$ and $T \in \mathscr{W}$. Then

$$S <_r G_1 \otimes G_2 \quad \text{for some } G_1 \in \mathscr{U} \text{ and } G_2 \in \mathscr{V},$$

it follows from Lemma 3.9(a) that

$$S \otimes T <_r (G_1 \otimes G_2) \otimes T <_r (G_1 \otimes G_2) \otimes T^1.$$

Since $\mathscr{W}$ is monoidal, $T^1 \in \mathscr{W}$. Thus by Corollary 4.2, we have

$$\begin{aligned} (G_1 \otimes G_2) \otimes T^1 &\in \mathscr{LZ} \circ (\mathscr{U} \otimes (\mathscr{V} \otimes \mathscr{W})) \\ &= \mathscr{LZ} \circ \langle (\mathscr{LZ} \vee \mathscr{U}) \circ (\mathscr{V} \otimes \mathscr{W}) \rangle_{er} \quad \text{by Result 3.6} \\ &\subseteq \langle (\mathscr{LZ} \circ (\mathscr{LZ} \vee \mathscr{U})) \circ (\mathscr{V} \otimes \mathscr{W}) \rangle_{er} \quad \text{by Result 2.4} \\ &= \langle (\mathscr{LZ} \vee \mathscr{U}) \circ (\mathscr{V} \otimes \mathscr{W}) \rangle_{er} \\ &= \mathscr{U} \otimes (\mathscr{V} \otimes \mathscr{W}) \quad \text{by Result 3.6,} \end{aligned}$$

it follows that $S \otimes T \in \mathscr{U} \otimes (\mathscr{V} \otimes \mathscr{W})$ and hence $(\mathscr{U} \otimes \mathscr{V}) \otimes \mathscr{W} \subseteq \mathscr{U} \otimes (\mathscr{V} \otimes \mathscr{W})$. Therefore, $\mathscr{U} \otimes (\mathscr{V} \otimes \mathscr{W}) = (\mathscr{U} \otimes \mathscr{V}) \otimes \mathscr{W}$. □

Lemma 4.6. *Let $Y \in \mathscr{S}$ be a monoid, $T \in \mathscr{RS}$ and $G \in \mathscr{G}$.*

(a) *There is a homomorphism*

$$\tau : (Y \otimes T)^* \otimes G \to (Y \otimes (T \otimes G))^*$$

such that $\bar{\tau}$ is over $\mathscr{LZ}$.

(b) $(Y \otimes (T \otimes G))^* <_r (Y \otimes T^G)^* \otimes G$.

Proof. (a) Similar to the proof of Lemma 4.1(a), for each $(f, r) \in (Y \otimes T)^* \otimes G$ with $xf = (\beta_x, t_x)$ for all $x \in G$, we let $(f, r)\tau = (\varphi, (\psi, r))$, where $\psi : G \to T$ is defined by $x\psi = t_x$ for all $x \in G$, and $\varphi : (T \otimes G)^1 \to Y$ is defined by $(p,x)\varphi = (1p)\beta_x$ for all $(p,x) \in T \otimes G$, and $1\varphi = 1\beta_1$. Clearly τ is well defined.

We first show that τ maps $(Y \otimes T)^* \otimes G$ into $(Y \otimes (T \otimes G))^*$. To show that $(f,r)\tau = (\varphi,(\psi,r)) \in (Y \otimes (T \otimes G))^*$, we let $(p,x) \in T \otimes G$ be such that $(p,x)\mathscr{R}(p,x)(\psi,r)$ in $T \otimes G$. Then $(p,x)\mathscr{R}(p^x\psi, xr)$, it follows easily that $(1p)\mathscr{R}(1p)t_x$ in T. Since $(\beta_x, t_x) \in (Y \otimes T)^*$, $(1p)\beta_x = 1$ so that $(p,x)\varphi = (1p)\beta_x = 1$. Hence $(\varphi,(\psi,r)) \in (Y \otimes (T \otimes G))^*$, as required.

The same argument as in the proof of Lemma 4.1(a) now yields that τ is a homomorphism.

We finally show that $\bar{\tau}$ is over $\mathscr{L}\mathscr{Z}$. So let $(f,r),(f_1,r_1) \in (Y \otimes T)^* \otimes G$ with $xf = (\beta_x, t_x)$ and $xf_1 = (\gamma_x, s_x)$ for all $x \in G$, be such that $(f,r)\tau = (f_1,r_1)\tau = (\varphi,(\psi,r)) = (\varphi_1,(\psi_1,r_1))$ is an idempotent in $(Y \otimes (T \otimes G))^*$. Then $\varphi = \varphi_1$, $\psi = \psi_1$ and $r = r_1$. Since $(\varphi,(\psi,r))$ is an idempotent, we have $\psi = \psi^1\psi$ and $r = 1$, the identity of G. From this it follows that $t_x = x\psi = x(\psi^1\psi) = (x\psi)(x\psi_1) = t_x s_x$ for all $x \in G$. Thus $(yt_x)\gamma_x = 1$ for all $y \in T^1$, since $(\gamma_x, s_x) \in (Y \otimes T)^*$. Hence $\beta_x = \beta_x{}^{t_x}\gamma_x$ for all $x \in G$, and so $(f,1)(f_1,1) = (f,1)$. Therefore $(\varphi,(\psi,1))\tau^{-1}$ is a left zero semigroup and $\bar{\tau}$ is over $\mathscr{L}\mathscr{Z}$.

(b) Similar to the proof of Lemma 4.3(a), for each $(\varphi,(\psi,r)) \in (Y \otimes (T \otimes G))^*$, we let

$$(\varphi,(\psi,r))\tau_1 = \left\{ (f,r) \in (Y \otimes T^G)^* \otimes G \;\middle|\; \begin{array}{l} \text{(i) } h_x = {}^x\psi; \\ \text{(ii) } (p,x)\varphi = p\beta_x \text{ and} \\ \text{(iii) } 1\varphi = 1\beta_1 \end{array} \right\},$$

where for each $x \in G$, $xf = (\beta_x, h_x) \in (Y \otimes T^G)^*$. Since $p\mathscr{R}ph_x$ in T^G implies that $(p,x)\mathscr{R}(p,x)(\psi,r)$ in $T \otimes G$, it follows easily that $(\varphi,(\psi,r))\tau_1 \neq \emptyset$. The argument of Lemma 4.3(a) applies to yield that τ_1 is also an injective relational morphism.

We now show that τ_1 is regular. So let $(f,r) \in (\varphi,(\psi,r))\tau_1$. Define $\psi_1 : G \to T$ by $y\psi_1 = (yr^{-1}\psi)'$ for some $(yr^{-1}\psi)' \in V(yr^{-1}\psi)$ in T, and define $\varphi_1 : (T \otimes G)^1 \to Y$ by $p\varphi_1 = (p(\psi_1, r^{-1}))\varphi$ for all $p \in (T \otimes G)^1$. We claim that $(\varphi_1,(\psi_1,r^{-1})) \in V((\varphi,(\psi,r)))$ in $Y \otimes (T \otimes G)$. Routine calculations show that $(\psi_1, r^{-1}) \in V((\psi,r))$ in $T \otimes G$. From this it follows that $(p(\psi,r)(\psi_1,r^{-1}))\varphi = 1$ for all $p \in (T \otimes G)^1$, since $(\varphi,(\psi,r)) \in (Y \otimes (T \otimes G))^*$. Thus for any $p \in (T \otimes G)^1$,

$$\begin{aligned} p(\varphi^{(\psi,r)}\varphi_1{}^{(\psi,r)(\psi_1,r^{-1})}\varphi) &= (p\varphi)(p(\psi,r))\varphi_1(p(\psi,r)(\psi_1,r^{-1}))\varphi \\ &= (p\varphi)(p(\psi,r)(\psi_1,r^{-1}))\varphi \\ &= p\varphi, \end{aligned}$$

so that $(\varphi,(\psi,r))(\varphi_1,(\psi_1,r^{-1}))(\varphi,(\psi,r)) = (\varphi,(\psi,r))$. By symmetry, $(\varphi_1,(\psi_1,r^{-1})) \in V((\varphi,(\psi,r))$. Since $p(\psi_1,r^{-1}) \mathscr{R} p(\psi_1,r^{-1})(\psi,r)$ for any $p \in (T \otimes G)^1$, it follows that $p\varphi_1 = (p(\psi_1,r^{-1}))\varphi = 1$, whence $(\varphi_1,(\psi_1,r^{-1})) \in (Y \otimes (T \otimes G))^*$. For each $x \in G$, we let $xf_1 = (\gamma_x, g_x)$, where $g_x = {}^x\psi_1$ and $\gamma_x : (T^G)^1 \to Y$ is defined by $p\gamma_x = 1$, — the identity of Y, for all $p \in (T^G)^1$. It is straightforward to verify that $(f_1, r^{-1}) \in V((f,r))$

in $(Y \otimes T^G)^* \otimes G$ and $(f_1, r^{-1}) \in ((\varphi_1, (\psi_1, r^{-1}))\tau_1$. Hence, τ_1 is an injective regular relational morphism, and therefore $(Y \otimes (T \otimes G))^* <_r (Y \otimes T^G)^* \otimes G$. □

Theorem 4.7. *Let* $\mathscr{U} \in \mathscr{L}_{er}(\mathscr{E}\mathscr{S})$ *and* $\mathscr{V} \in \mathscr{L}_{er}(\mathscr{G})$. *Then*

$$\mathscr{S} \otimes^* (\mathscr{U} \otimes \mathscr{V}) = (\mathscr{S} \otimes^* \mathscr{U}) \otimes \mathscr{V}.$$

Proof. If $\mathscr{U} \subseteq \mathscr{G}$, then by Result 3.10, $\mathscr{S} \otimes^* (\mathscr{U} \otimes \mathscr{V}) = \mathscr{U} \otimes \mathscr{V} = (\mathscr{S} \otimes^* \mathscr{U}) \otimes \mathscr{V}$. So we assume that $\mathscr{U} \nsubseteq \mathscr{G}$. To show that $\mathscr{S} \otimes^* (\mathscr{U} \otimes \mathscr{V}) \subseteq (\mathscr{S} \otimes^* \mathscr{U}) \otimes \mathscr{V}$, let $Y \in \mathscr{S}$ be a monoid and $S \in \mathscr{U} \otimes \mathscr{V}$. By Results 3.2 and 3.3, $S <_r R \otimes G$ for some $R \in \mathscr{U}$ and $G \in \mathscr{V}$. It follows from Lemmas 3.12 and 4.6(b) that

$$(Y \otimes S)^* <_r (Y \otimes (R \otimes G))^* <_r (Y \otimes R^G)^* \otimes G,$$

where the latter belongs to $(\mathscr{S} \otimes^* \mathscr{U}) \otimes \mathscr{V}$. Thus $(Y \otimes S)^* \in (\mathscr{S} \otimes^* \mathscr{U}) \otimes \mathscr{V}$, and hence $\mathscr{S} \otimes^* (\mathscr{U} \otimes \mathscr{V}) \subseteq (\mathscr{S} \otimes^* \mathscr{U}) \otimes \mathscr{V}$.

For the converse, let $S \in \mathscr{S} \otimes^* \mathscr{U}$ and $G \in \mathscr{V}$. It follows from Lemma 3.11 that $S <_r (Y \otimes T)^*$ for some monoid $Y \in \mathscr{S}$ and $T \in \mathscr{U}$. Thus, by Lemma 3.8, $S \otimes G <_r (Y \otimes T)^* \otimes G$ so that by Lemma 4.6(a)

$$\begin{aligned} S \otimes G &\in \mathscr{L}\mathscr{Z} \circ (\mathscr{S} \otimes^* (\mathscr{U} \otimes \mathscr{V})) \\ &\subseteq \mathscr{L}\mathscr{Z} \circ (\mathscr{L}\mathscr{Z} \circ (\mathscr{U} \otimes \mathscr{V})) \quad \text{by Result 3.10} \\ &\subseteq (\mathscr{L}\mathscr{Z} \circ \mathscr{L}\mathscr{Z}) \circ (\mathscr{U} \otimes \mathscr{V}) \quad \text{by Result 2.4} \\ &= \mathscr{L}\mathscr{Z} \circ (\mathscr{U} \otimes \mathscr{V}) \\ &= \mathscr{S} \otimes^* (\mathscr{U} \otimes \mathscr{V}). \end{aligned}$$

It follows that $(\mathscr{S} \otimes^* \mathscr{U}) \otimes \mathscr{V} \subseteq \mathscr{S} \otimes^* (\mathscr{U} \otimes \mathscr{V})$, as required. □

We conclude this section with the following useful observations.

Proposition 4.8. *For any* $\mathscr{U} \in \mathscr{L}_{er}(\mathscr{I}L)$ *and* $\mathscr{V} \in \mathscr{L}_{er}(\mathscr{G})$, *we have*

$$\mathscr{G} \otimes (\mathscr{U} \otimes \mathscr{V}) \subseteq (\mathscr{G} \otimes \mathscr{U}) \otimes \mathscr{V}$$

and

$$\mathscr{S} \otimes^* (\mathscr{U} \otimes \mathscr{V}) \subseteq (\mathscr{S} \otimes^* \mathscr{U}) \otimes \mathscr{V}.$$

Proof. This follows immediately from the proofs of Theorems 4.4(a) and 4.7. □

5. Some sufficient conditions for $\mathscr{U} \otimes \mathscr{G} = \mathscr{U} C_\infty$

Let $S \in \mathscr{R}\mathscr{S}$. A subsemigroup R of S is *full* if $E(S) \subseteq R$; R is *self-conjugate* if $a'Ra \subseteq R$ for each $a \in S$ and $a' \in V(a)$. We denote by R_c the subsemigroup of S

generated by the conjugates of R in S^1; that is

$$R_c = \langle a'ra \mid r \in R,\ a \in S^1,\ a' \in V(a) \rangle.$$

Let $C_\infty(S)$ denote the *least full and self-conjugate subsemigroup* of S. So

$$C_\infty(S) = C(S)_{ccc} \ldots$$

By [31, Lemma 2.3], $C_\infty(S)$ is a full regular subsemigroup of S.

For any $\mathscr{U} \in \mathscr{L}_{ev}(\mathscr{RS})$, let

$$\mathscr{U}C_\infty = \{S \in \mathscr{RS} \mid C_\infty(S) \in \mathscr{U}\}.$$

Then $\mathscr{U}C_\infty$ is again an e-variety. This operator was introduced in [12, 25]. Its restriction to $\mathscr{L}(\mathscr{CR})$ was studied in [34, 36], where the notation C^* is used.

Result 5.1 (Jones and Trotter [12], and Reilly and Zhang [25]). *For any* $\mathscr{U} \in \mathscr{L}_{ev}(\mathscr{RS})$, $\mathscr{U}C_\infty = \langle \mathscr{U} \circ \mathscr{G} \rangle_{ev}$.

Clearly, $\mathscr{U}C_\infty \subseteq \mathscr{U}C$ for any $\mathscr{U} \in \mathscr{L}_{ev}(\mathscr{RS})$. It follows from [31, Corollary 5.2] that $\mathscr{ES} = \mathscr{CR}C = \mathscr{CR}C_\infty$.

Result 5.2 (Reilly and Zhang [26]). C_∞ *commutes with* T_ℓ, T_r, K_ℓ *and* K_r *on* $\mathscr{L}_{ev}(\mathscr{RS})$.

Let $\mathscr{U} \in \mathscr{L}_{ev}(\mathscr{RX})$. By Result 3.3, Lemma 3.7 and Result 5.1, we have

$$\mathscr{U} \otimes \mathscr{G} = \mathscr{U} * \mathscr{G} \subseteq \langle \mathscr{U} \circ \mathscr{G} \rangle_{ev} = \mathscr{U}C_\infty.$$

In this section we first present a family of examples to show that the equality $\mathscr{U} \otimes \mathscr{G} = \mathscr{U}C_\infty$ need not hold in general. We then present some sufficient conditions for $\mathscr{U} \otimes \mathscr{G}$ to be equal to $\langle \mathscr{U} \circ \mathscr{G} \rangle_{ev}$. The next result was essentially proved in [14] (see also [12]).

Result 5.3 (Jones and Trotter [12]). *Let* $\mathscr{U} \in \mathscr{L}_{ev}(\mathscr{C})$. *Then*

$$\mathscr{U} \otimes \mathscr{G} = \mathscr{U} * \mathscr{G} = \mathscr{U}C_\infty = \mathscr{U}C.$$

The next example presents an infinite family of e-varieties of completely simple semigroups with the property that $\mathscr{U} \otimes \mathscr{G} \neq \mathscr{U}C_\infty$.

Example 5.4. There exits an infinite family of e-varieties of completely simple semigroups:

$$\mathscr{V}_0 \subset \mathscr{V}_1 \subset \mathscr{V}_2 \subset \cdots \subset \mathscr{V}_i \subset \cdots$$

such that $\mathscr{V}_i \otimes \mathscr{G} \neq \langle \mathscr{V}_i \circ \mathscr{G} \rangle = \mathscr{V}_i C_\infty$ for all $i \geq 0$.

Proof. Let $\mathscr{U}_0 = \mathscr{A}\mathscr{G} \circ \mathscr{R}\mathscr{B}$, and for each $i \geq 1$, we let $\mathscr{U}_i = \mathscr{U}_{i-1}C_\infty$. Thus $\mathscr{U}_i = \mathscr{U}_0 (C_\infty)^i$ for $i \geq 0$. By [36, Proposition 3.7], the e-varieties $\mathscr{U}_i$, $i \geq 0$, form a strictly ascending sequence

$$\mathscr{U}_0 \subset \mathscr{U}_1 \subset \mathscr{U}_2 \subset \cdots \subset \mathscr{U}_i \subset \cdots \tag{6}$$

such that $\bigvee_{i\geq 0} \mathscr{U}_i \subset \mathscr{U}_0 C \subset \mathscr{C}\mathscr{S}$. For each $i \geq 0$, let $\mathscr{V}_i = \mathscr{U}_i \otimes \mathscr{G}$. We now claim that $\mathscr{V}_i \otimes \mathscr{G} \neq \langle \mathscr{V}_i \circ \mathscr{G} \rangle = \mathscr{V}_i C_\infty$ for all $i \geq 0$. Assume that $\mathscr{V}_i \otimes \mathscr{G} = \mathscr{V}_i C_\infty$ for some $i \geq 0$. Then

$$\begin{aligned}\mathscr{V}_i \otimes \mathscr{G} &= (\mathscr{U}_i \otimes \mathscr{G}) \otimes \mathscr{G} \\ &= \mathscr{U}_i \otimes (\mathscr{G} \otimes \mathscr{G}) \quad \text{by Theorem 4.4} \\ &= \mathscr{U}_i \otimes \mathscr{G} \\ &= \mathscr{V}_i,\end{aligned}$$

so that $\mathscr{V}_i C_\infty = \mathscr{V}_i$ and $\mathscr{V}_i(C_\infty)^2 = \mathscr{V}_i$. On the other hand, we have

$$\mathscr{V}_i = \mathscr{U}_i \otimes \mathscr{G} \subseteq \mathscr{U}_i C_\infty = \mathscr{U}_{i+1}$$

and

$$\mathscr{U}_{i+2} = \mathscr{U}_i(C_\infty)^2 \subseteq \mathscr{V}_i(C_\infty)^2 = \mathscr{V}_i \subset \mathscr{U}_{i+1},$$

from which it follows that $\mathscr{U}_{i+1} = \mathscr{U}_{i+2}$, and by (6), this is a contradiction. Hence $\mathscr{V}_i \otimes \mathscr{G} \neq \mathscr{V}_i C_\infty$ for all $i \geq 0$. From this it also follows that the e-varieties $\mathscr{V}_i$, $i \geq 0$, form a strictly ascending sequence

$$\mathscr{V}_0 \subset \mathscr{V}_1 \subset \mathscr{V}_2 \subset \cdots \subset \mathscr{V}_i \subset \cdots \tag{7}$$

such that $\bigvee_{i\geq 0} \mathscr{V}_i = \bigvee_{i\geq 0} \mathscr{U}_i \subset \mathscr{U}_0 C \subset \mathscr{C}\mathscr{S}$. □

We now prove the main result of this section.

Theorem 5.5. *Let* $\mathscr{U} \in \mathscr{L}_{ev}(\mathscr{E}\mathscr{S}) \cup \mathscr{L}_{ev}(\mathscr{I}L)$ *be such that* $\mathscr{U} \otimes \mathscr{G} = \mathscr{U}C_\infty$. *Then* $(\mathscr{U}^P) \otimes \mathscr{G} = (\mathscr{U}^P)C_\infty$ *for any* $P \in \{T_\ell, T_r, K_\ell, K_r\}$.

Proof. Consider T_ℓ. If $\mathscr{U} \subseteq \mathscr{G}$, then $\mathscr{U}^{T_\ell} \in \mathscr{L}_{ev}(\mathscr{C})$ and the equality follows from Result 5.3. So we assume that $\mathscr{U} \not\subseteq \mathscr{G}$. There are two cases:

(a) $\mathscr{U} \subseteq \mathscr{E}\mathscr{S}$: Then

$$\begin{aligned}(\mathscr{U}^{T_\ell})C_\infty &= (\mathscr{U}C_\infty)^{T_\ell} \quad \text{by Result 5.2} \\ &= \mathscr{G} \otimes (\mathscr{U} \otimes \mathscr{G}) \quad \text{by the hypothesis and Result 3.6} \\ &= (\mathscr{G} \otimes \mathscr{U}) \otimes \mathscr{G} \quad \text{by Theorem 4.4} \\ &= (\mathscr{U}^{T_\ell}) \otimes \mathscr{G},\end{aligned}$$

as required.

(b) $\mathscr{U} \subseteq \mathscr{I}L$: Then

$$\begin{aligned}
(\mathscr{U}^{T_l})C_\infty &= (\mathscr{U}C_\infty)^{T_l} \quad \text{by Result 5.2} \\
&= \mathscr{G} \otimes (\mathscr{U} \otimes \mathscr{G}) \\
&\subseteq (\mathscr{G} \otimes \mathscr{U}) \otimes \mathscr{G} \quad \text{by Proposition 4.8} \\
&= (\mathscr{U}^{T_l}) \otimes \mathscr{G} \\
&\subseteq \langle \mathscr{U}^{T_l} \circ \mathscr{G} \rangle_{er} \quad \text{by Lemma 3.7(a)} \\
&= (\mathscr{U}^{T_l})C_\infty \quad \text{by Result 5.1.}
\end{aligned}$$

It follows that $(\mathscr{U}^{T_l})C_\infty = (\mathscr{U}^{T_l}) \otimes \mathscr{G}$.

The equality for T_r follows similarly by using Lemma 3.7(b) and the duals of Result 3.6 and Theorem 4.4. The same argument as for T_l and T_r also yields the equalities for K_l and K_r using Result 3.10, Theorem 4.7, Proposition 4.8 and their duals. □

Remark 5.6. Let $T = T_l \cap T_r$. Then by [24, Corollary 6.3], $\mathscr{U}^T = \mathscr{G} \circ \mathscr{U}$ for all $\mathscr{U} \in \mathscr{L}_{er}(\mathscr{R}\mathscr{S})$. Whether Theorem 5.5 holds for T is still open. In particular, we do not know whether or not $\mathscr{B}\mathscr{G} \otimes \mathscr{G} = (\mathscr{B}\mathscr{G})C_\infty$, where $\mathscr{B}\mathscr{G} = \mathscr{G} \circ \mathscr{B}$ is the e-variety of all bands of groups.

As an analogue to the notion of *locality* of monoid varieties in the sense of Tilson [30], Szendrei [28] introduced the notion of *e-locality* (or *bi-locality*) of e-varieties. We shall not go into this topic in detail, and refer the readers to [11, 12, 28] for definitions and detailed information. Using this notion, Jones and Trotter [12] gave another sufficient condition for $\mathscr{U} \otimes \mathscr{G} = \mathscr{U}C_\infty$ as follows.

Result 5.7 (Jones and Trotter [12]). *If $\mathscr{U} \in \mathscr{L}_{er}(\mathscr{E}\mathscr{S})$ is e-local, then $\mathscr{U} \otimes \mathscr{G} = \mathscr{U}C_\infty$.*

In [28], Szendrei showed that the e-variety of all orthodox semigroups is e-local, and every e-variety of orthogroups is e-local. In particular, every e-variety of bands is e-local. Jones [11] showed that for each variety $\mathscr{H}$ of groups the e-variety $\mathscr{C}\mathscr{R}(\mathscr{H})$, consisting of all completely regular semigroups whose subgroups belong to $\mathscr{H}$, is e-local. In particular, $\mathscr{C}\mathscr{R}$ is e-local. As the next remark points out, not every e-variety of regular semigroups is e-local.

Remark 5.8. (This observation was pointed out to the authors by P.R. Jones.) It follows from Result 5.7 that Example 5.4 presents an infinite family of e-varieties of completely simple semigroups which are not e-local.

For any variety $\mathscr{H}$ of groups, we denote by $\hat{\mathscr{H}}$ the largest variety of inverse semigroups having E-unitary covers over $\mathscr{H}$ (see [21, Definition XII.9.3]).

Result 5.9 (Petrich [21]). *Let* $\mathscr{H} \in \mathscr{L}(\mathscr{G})$. *Then*

$$\begin{aligned}\hat{\mathscr{H}} &= \langle \mathscr{S} \circ \mathscr{H} \rangle_{er} \\ &= [u^2 = u \mid u = 1 \textit{ is a law in } \mathscr{H}].\end{aligned}$$

Result 5.10 (Reilly and Zhang [26]). *Let* $\mathscr{H} \in \mathscr{L}(\mathscr{G})$. *Then*
(a) $(\hat{\mathscr{H}})^M = \mathscr{C}\mathscr{S} \circ \hat{\mathscr{H}} = \bigvee_{n \geq 0} (\hat{\mathscr{H}})^{(T_l T_r)^n}$.
(b) $(\hat{\mathscr{H}})^K = \mathscr{R}\mathscr{B} \circ \hat{\mathscr{H}} = \bigvee_{n \geq 0} (\hat{\mathscr{H}})^{(K_l K_r)^n}$.

The equalities $\mathscr{C}\mathscr{R} \otimes \mathscr{H} = \langle \mathscr{C}\mathscr{R} \circ \mathscr{H} \rangle_{er}$ and $\mathscr{B} \otimes \mathscr{H} = \langle \mathscr{B} \circ \mathscr{H} \rangle_{er}$ for any group variety $\mathscr{H}$ in the next result were proved by Jones and Trotter [12]. For the case $\mathscr{H} = \mathscr{G}$, the equality $\mathscr{E}\mathscr{S} = \mathscr{C}\mathscr{R} \otimes \mathscr{G}$ was first obtained by Szendrei [29] as a consequence of her description of the bifree regular E-solid semigroup.

Theorem 5.11. *Let* $\mathscr{H} \in \mathscr{L}_{er}(G)$. *Then*
(a) $(\hat{\mathscr{H}})^M = \langle \mathscr{C}\mathscr{R} \circ \mathscr{H} \rangle_{er} = \mathscr{C}\mathscr{R} \otimes \mathscr{H}$.
(b) $(\hat{\mathscr{H}})^K = \langle \mathscr{B} \circ \mathscr{H} \rangle_{er} = \mathscr{B} \otimes \mathscr{H}$.

Proof. (a) Since $\mathscr{S}$ is e-local (see [28]), it follows from [12, Proposition 5.1] that $\hat{\mathscr{H}} = \mathscr{S} \otimes \mathscr{H}$. Using the fact that $\mathscr{C}\mathscr{R} = \bigvee_{n \geq 0} \mathscr{S}^{(T_l T_r)^n}$, we have

$$\begin{aligned}\mathscr{C}\mathscr{R} \otimes \mathscr{H} &= \left(\bigvee_{n \geq 0} \mathscr{S}^{(T_l T_r)^n} \right) \otimes \mathscr{H} \\ &= \bigvee_{n \geq 0} \mathscr{S}^{(T_l T_r)^n} \otimes \mathscr{H} \quad \text{by Proposition 3.13(c)} \\ &= \bigvee_{n \geq 0} (\mathscr{S} \otimes \mathscr{H})^{(T_l T_r)^n} \quad \text{by Result 3.6, Theorem 4.4 and their duals} \\ &= \bigvee_{n \geq 0} (\hat{\mathscr{H}})^{(T_l T_r)^n} \\ &= (\hat{\mathscr{H}})^M \quad \text{by Result 5.10(a),}\end{aligned}$$

as required.
(b) This follows similarly by using the fact that $\mathscr{B} = \bigvee_{n \geq 0} \mathscr{S}^{(K_l K_r)^n}$. □

In some sense we have reduced the *membership problem* of $\mathscr{C}\mathscr{R} \otimes \mathscr{H}$ to that of $\hat{\mathscr{H}}$.

6. E-varieties of the form $\mathcal{CS}\otimes\mathcal{U}$

Our goal in this section is to describe the e-varieties of the form $\mathcal{U}\otimes\mathcal{V}$ with $\mathcal{U}\subseteq\mathcal{CS}$. We first consider the cases when $\mathcal{U}\in\{\mathcal{LZ},\mathcal{RZ},\mathcal{RB}\}$.

Lemma 6.1. *For any $\mathcal{V}\in\mathcal{L}_{er}(\mathcal{RS})$, we have $\mathcal{LZ}\otimes\mathcal{V}=\mathcal{LZ}\vee\mathcal{V}$.*

Proof. It is clear that $\mathcal{LZ}\vee\mathcal{V}\subseteq\mathcal{LZ}\otimes\mathcal{V}$. For the converse, let $L\in\mathcal{LZ}$ and $T\in\mathcal{V}$. Since the projection $\pi_1:\mathrm{Reg}(L\otimes T)\to T$ is a homomorphism, it follows that $\mathrm{Reg}(L\otimes T)/\bar{\pi}_1\in\mathcal{V}$. On the other hand, $L\in\mathcal{LZ}$ implies that the projection $\pi_2:\mathrm{Reg}(L\otimes T)\to L^{T^1}$ is also a homomorphism, so that $\mathrm{Reg}(L\otimes T)/\bar{\pi}_2\in\mathcal{LZ}$. But clearly $\bar{\pi}_1\cap\bar{\pi}_2=\iota$ on $\mathrm{Reg}(L\otimes T)$. It follows that $\mathrm{Reg}(L\otimes T)$ is a subdirect product of $\mathrm{Reg}(L\otimes T)/\bar{\pi}_2$ and $\mathrm{Reg}(L\otimes T)/\bar{\pi}_1$, so that $\mathrm{Reg}(L\otimes T)\in\mathcal{LZ}\vee\mathcal{V}$. Hence, $\mathcal{LZ}\otimes\mathcal{V}\subseteq\mathcal{LZ}\vee\mathcal{V}$, as required. □

Lemma 6.2. *For any $\mathcal{V}\in\mathcal{L}_{er}(\mathcal{RS})$, we have $\mathcal{RZ}\otimes\mathcal{V}\subseteq\mathcal{RZ}\circ\mathcal{V}=\mathcal{V}^{K_r}$.*

Proof. Let $\mathcal{V}\in\mathcal{L}_{er}(\mathcal{RS})$. Since $\mathcal{RZ}\circ\mathcal{V}$ is an e-variety, it suffices to show that $\mathrm{Reg}(R\otimes T)\in\mathcal{RZ}\circ\mathcal{V}$ for any $R\in\mathcal{RZ}$ and $T\in\mathcal{V}$. So let $R\in\mathcal{RZ}$ and $T\in\mathcal{V}$. First observe that if $(f,e)\in\mathrm{Reg}(R\otimes T)$ with $e\in E(T)$, then ${}^{e}f=f$, that is, (f,e) is an idempotent. Since the projection $\pi:\mathrm{Reg}(R\otimes T)\to T$ is a homomorphism, it follows that $\mathrm{Reg}(R\otimes T)/\bar{\pi}\in\mathcal{V}$. It remains to show that $\bar{\pi}$ is over $\mathcal{RZ}$. So let $(f,t),(g,t)\in\mathrm{Reg}(R\otimes T)$ be such that $(f,t)\pi=(g,t)\pi=t\in E(T)$. Then by the above observation, we have

$$(f,t)(g,t)=(f\,{}^{t}g,t^2)=({}^{t}g,t)=(g,t),$$

so that $t\pi^{-1}$ is a right zero semigroup. Hence, $\bar{\pi}$ is over $\mathcal{RZ}$, and therefore $\mathrm{Reg}(R\otimes T)\in\mathcal{RZ}\circ\mathcal{V}$, as required. □

Corollary 6.3. *For any $\mathcal{V}\in\mathcal{L}_{er}(\mathcal{RS})$, $\mathcal{RZ}\otimes\mathcal{V}^{K_r}=\mathcal{V}^{K_r}$. In particular, if $\mathcal{V}\in\{\mathcal{B},\mathcal{CS},\mathcal{CR},\mathcal{C},\mathcal{ES}\}$, then $\mathcal{RZ}\otimes\mathcal{V}=\mathcal{V}$.*

Proof. This is an immediate consequence of Lemma 6.2. □

As a corollary of [12, Corollary 3.7], we have the following.

Result 6.4. *For any $\mathcal{V}\in\mathcal{L}_{er}(\mathcal{RS})$, $\mathcal{RZ}\otimes(\mathcal{V}L)=\mathcal{V}L$. In particular,*

(i) $\mathcal{RZ}\otimes(\mathcal{B}L)=\mathcal{B}L$;
(ii) $\mathcal{RZ}\otimes(\mathcal{CR}L)=\mathcal{CR}L$;
(iii) $\mathcal{RZ}\otimes(\mathcal{I}L)=\mathcal{I}L$;
(iv) $\mathcal{RZ}\otimes(\mathcal{C}L)=\mathcal{C}L$;
(v) $\mathcal{RZ}\otimes(\mathcal{ES}L)=\mathcal{ES}L$.

It is well-known that $\mathscr{CR}$ also forms a variety of unary semigroups, determined by the identities $xx^{-1}x=x$, $(x^{-1})^{-1}=x$ and $xx^{-1}=x^{-1}x$. There have been great advances in the study of the lattice $\mathscr{L}(\mathscr{CR})$ of subvarieties of $\mathscr{CR}$ in recent years. For an extensive bibliography, consult Petrich and Reilly [22]. Since any regular subsemigroup of a completely regular semigroup is also completely regular, it follows that $\mathscr{L}(\mathscr{CR})$ is a complete sublattice of $\mathscr{L}_{ev}(\mathscr{RS})$ and that each e-subvariety of $\mathscr{CR}$ consists of completely regular semigroups.

Proposition 6.5. *For any* $\mathscr{V}\in\mathscr{L}_{ev}(\mathscr{CR})$, *we have* $\mathscr{RZ}\otimes\mathscr{V}=\mathscr{RZ}\vee\mathscr{V}$.

Proof. Let $\mathscr{V}\in\mathscr{L}_{ev}(\mathscr{CR})$. Then $\mathscr{V}\in\mathscr{L}(\mathscr{CR})$ and by Birkhoff's Theorem, there exists a set of (unary semigroup) identities $\{u_\alpha=v_\alpha\}_{\alpha\in A}$ such that $\mathscr{V}=[u_\alpha=v_\alpha]_{\alpha\in A}$. By [35, Proposition 2.9], we have $\mathscr{RZ}\vee\mathscr{V}=[u_\alpha x=v_\alpha x]_{\alpha\in A}$, where $x\notin c(u_\alpha)\cup c(v_\alpha)$. To show that $\mathscr{RZ}\otimes\mathscr{V}=\mathscr{RZ}\vee\mathscr{V}$, it suffices to show that $\mathscr{RZ}\otimes\mathscr{V}\subseteq[u_\alpha x=v_\alpha x]_{\alpha\in A}$. For each $\alpha\in A$, let $u_\alpha=u_\alpha(x_1,\ldots,x_n)=u_\alpha(x_i)$ and $v_\alpha=v_\alpha(x_1,\ldots,x_n)=v_\alpha(x_i)$. Now let $R\in\mathscr{RZ}$, $T\in\mathscr{V}$, $(f_i,t_i),(f,t)\in$ Reg$(R\otimes T)$, for $1\le i\le n$. For $\alpha\in A$, we then have

$$\begin{aligned}u_\alpha((f_i,t_i))(f,t)&=(g,u_\alpha(t_i))(f,t)\quad\text{for some } g\in R^{T^1}\\&=(g^{u_\alpha(t_i)}f,\ u_\alpha(t_i)t)\\&=({}^{u_\alpha(t_i)}f,\ u_\alpha(t_i)t)\quad\text{since } R\in\mathscr{RZ}\end{aligned}$$

and

$$\begin{aligned}v_\alpha((f_i,t_i))(f,t)&=(h,\ v_\alpha(t_i))(f,t)\quad\text{for some } h\in R^{T^1}\\&=(h^{v_\alpha(t_i)}f,v_\alpha(t_i)t)\\&=({}^{v_\alpha(t_i)}f,\ v_\alpha(t_i)t).\end{aligned}$$

Since $T\in\mathscr{V}=[u_\alpha(x_i)=v_\alpha(x_i)]$, it follows that $u_\alpha(t_i)=v_\alpha(t_i)$ and $u_\alpha(t_i)t=v_\alpha(t_i)t$. Thus, $u_\alpha((f_i,t_i))(f,t)=v_\alpha((f_i,t_i))(f,t)$ and hence Reg$(R\otimes T)$ satisfies the identity $u_\alpha(x_i)x=v_\alpha(x_i)x$. Consequently, Reg$(R\otimes T)\in[u_\alpha x=v_\alpha x]_{\alpha\in A}=\mathscr{RZ}\vee\mathscr{V}$, and therefore $\mathscr{RZ}\otimes\mathscr{V}\subseteq\mathscr{RZ}\vee\mathscr{V}$, as required. □

In [7], Hall established a Birkhoff-type theorem for e-varieties of regular semigroups by showing that each e-variety of regular semigroups is determined by a set of unary semigroup identities. In the context of orthodox semigroups, Kadourek and Szendrei [14] introduced the notion of biidentity, and associated with each e-variety $\mathscr{U}$ of orthodox semigroups the set of all biidentities which hold in $\mathscr{U}$. Similar results (to the orthodox case in [14]) for e-varieties of locally inverse semigroups or E-solid regular semigroups have been obtained by Auinger [1]. Here we shall use a similar theory developed by Kadourek [13] for locally orthodox semigroups.

As in [13], let X be a nonempty set, let $X'=\{x'\mid x\in X\}$ be a disjoint copy of X, and let $I=X\cup X'$. In addition, let $I\wedge I=\{(x,y)\in I\times I\mid x\neq y'$ or $y\neq x'\}$. We shall write $(x\wedge y)$ for (x,y) in $I\wedge I$. Let $F'^{\wedge}(X)$ denote the absolutely free semigroup on the set $I\cup I\wedge I$. By a *triidentity* over X we mean any pair $u=v$ of words $u,v\in F'^{\wedge}(X)$.

Let $S \in \mathcal{CL}$. A mapping $\theta : I \cup I \wedge I \to S$ is called *tied* if the following two conditions hold:

(i) $x'\theta \in V(x\theta)$ for any $x \in X$, and

(ii) $(x \wedge y)\theta \in S(y\theta, x\theta)$ for any $(x \wedge y) \in I \wedge I$.

We say that a triidentity $u = v$ is *satisfied* in S if, for any tied mapping $\theta : I \cup I \wedge I \to S$, we have $u\bar{\theta} = v\bar{\theta}$ where $\bar{\theta} : F'^{\wedge}(X) \to S$ is the unique homomorphism extending θ. The triidentity $u = v$ is *satisfied* in a class $\mathcal{V}$ of locally orthodox semigroups if it is satisfied in each member of $\mathcal{V}$.

For any set Σ of triidentities over X, let $[\Sigma]$ be the class of all locally orthodox semigroups in which all triidentities are satisfied. As pointed out in [13], $[\Sigma]$ need not be an e-variety in general. However, the following result will be useful.

Result 6.6 (Kadourek [13]). *If $\mathcal{V} \in \mathcal{L}_{er}(\mathcal{CL})$, then there exists a set Σ of triidentities such that $\mathcal{V} = [\Sigma]$.*

For any $u \in F'^{\wedge}(X)$ we let

$c(u)$ – the set of all elements of $I \cup I \wedge I$ that appears in u,

$t(u)$ – the *tail* of u, that is, the element of $I \cup I \wedge I$ occurring last from the left in u.

We are now ready to prove our next result.

Proposition 6.7. *For any $\mathcal{V} \in \mathcal{L}_{er}(\mathcal{C}L)$, $\mathcal{RZ} \otimes \mathcal{V} = \mathcal{RZ} \vee \mathcal{V}$.*

Proof. Let $\mathcal{V} \in \mathcal{L}_{er}(\mathcal{C}L)$. Note that $\mathcal{RZ} \vee \mathcal{V} \subseteq \mathcal{RZ} \otimes \mathcal{V} \subseteq \mathcal{RZ} \otimes (\mathcal{RZ} \vee \mathcal{V})$. To show that $\mathcal{RZ} \otimes \mathcal{V} = \mathcal{RZ} \vee \mathcal{V}$, it suffices to show that $\mathcal{RZ} \otimes (\mathcal{RZ} \vee \mathcal{V}) \subseteq \mathcal{RZ} \vee \mathcal{V}$. By Result 6.6, there exists a set $\Sigma = \{u_\alpha = v_\alpha\}_{\alpha \in A}$ of triidentities such that $\mathcal{RZ} \vee \mathcal{V} = [u_\alpha = v_\alpha]_{\alpha \in A}$. By Corollary 6.4, $\mathcal{RZ} \otimes (\mathcal{RZ} \vee \mathcal{V}) \subseteq \mathcal{C}L$. The same type of argument as in the proof of Proposition 6.5 yields that $\mathcal{RZ} \otimes (\mathcal{RZ} \vee \mathcal{V}) \subseteq [u_\alpha z = v_\alpha z]_{\alpha \in A}$, where $z \notin c(u_\alpha) \cup c(v_\alpha)$ for all $\alpha \in A$. Note that $S(a,b) = bV(ab)a$, for any $S \in \mathcal{RS}$ and $a, b \in S$. For each $\alpha \in A$, since $u_\alpha = v_\alpha$ is satisfied in $\mathcal{RZ}$, it follows that (i) $t(u_\alpha) = t(v_\alpha) = x$, for some $x \in X \cup X'$; or (ii) $t(u_\alpha) = (y \wedge x)$ and $t(v_\alpha) = (y_1 \wedge x)$, for some $x, y, y_1 \in X \cup X'$; or (iii) $t(u_\alpha) = x$ and $t(v_\alpha) = (y \wedge x)$, for some $x, y \in X \cup X'$; or (iv) $t(u_\alpha) = (y \wedge x)$ and $t(v_\alpha) = x$, for some $x, y \in X \cup X'$. In each of the above cases, by letting $z = x'x$, that $u_\alpha z = v_\alpha z$ is satisfied in $\mathcal{RZ} \otimes (\mathcal{RZ} \vee \mathcal{V})$ implies that $u_\alpha = v_\alpha$ is also satisfied in $\mathcal{RZ} \otimes (\mathcal{RZ} \vee \mathcal{V})$. It follows that $\mathcal{RZ} \otimes (\mathcal{RZ} \vee \mathcal{V}) \subseteq [u_\alpha = v_\alpha]_{\alpha \in A} = \mathcal{RZ} \vee \mathcal{V}$, as required. □

The above results automatically lead us to the following question:

Question 6.8. *Is it true that $\mathcal{RZ} \otimes \mathcal{V} = \mathcal{RZ} \vee \mathcal{V}$ for any $\mathcal{V} \in \mathcal{L}_{er}(\mathcal{RS})$?*

Combining Proposition 3.13(a) and Lemma 6.1 with the results of this section, we have the following.

Corollary 6.9. *For any* $\mathscr{V} \in \mathscr{L}_{er}(\mathscr{RS})$, $\mathscr{RB} \otimes \mathscr{V} = \mathscr{LZ} \vee \mathscr{RZ} \otimes \mathscr{V}$. *In particular, if* $\mathscr{V} \in \mathscr{L}_{er}(\mathscr{CR})$ *or* $\mathscr{L}_{er}(\mathscr{C}L)$, *then* $\mathscr{RB} \otimes \mathscr{V} = \mathscr{RB} \vee \mathscr{V}$.

Theorem 6.10. *Let* $\mathscr{U} \in \mathscr{L}_{er}(\mathscr{G})$, *and let* $\mathscr{V} \in \mathscr{L}_{er}(\mathscr{RS})$ *be monoidal. Then*

(a) $(\mathscr{U} \otimes \mathscr{RZ}) \otimes \mathscr{V} \subseteq \mathscr{U} \otimes (\mathscr{RZ} \otimes \mathscr{V})$,

(b) *if* $\mathscr{V} \in \mathscr{L}_{er}(\mathscr{ES})$ *or* $\mathscr{L}_{er}(\mathscr{I}L)$,

$$(\mathscr{U} \otimes \mathscr{RZ}) \otimes \mathscr{V} = \mathscr{U} \otimes (\mathscr{RZ} \otimes \mathscr{V}).$$

Proof. (a) If $\mathscr{U} = \mathscr{T}$, then the inclusion clearly holds. So we assume that $\mathscr{U} \neq \mathscr{T}$. Let $S \in \mathscr{U} \otimes \mathscr{RZ}$ and $R \in \mathscr{V}$. Since $\mathscr{V}$ is monoidal, $R^1 \in \mathscr{V}$. By Result 3.6, $\mathscr{U} \otimes \mathscr{RZ} \subseteq \mathscr{CS}$ so that by Ree's Theorem (see [9, Theorem III.2.11]), $S = \mathscr{M}(G; I, \Lambda; P)$ with $G \in \mathscr{U}$. It follows from [12, Proposition 4.1] that S embeds in $(G \otimes T) \times L$ for some $T \in \mathscr{RZ}$ and $L \in \mathscr{LZ}$. Thus we clearly have

$$\begin{aligned} \mathrm{Reg}(S \otimes R) &<_r \mathrm{Reg}(S \otimes R^1) \\ &<_r \mathrm{Reg}((G \otimes T) \times L) \otimes R^1) \quad \text{by Lemma 3.9(b)} \\ &<_r \mathrm{Reg}((G \otimes T) \otimes R^1) \times \mathrm{Reg}(L \otimes R^1), \end{aligned}$$

by a natural embedding. From the proof of Lemma 6.1 and Result 3.6,

$$\mathrm{Reg}(L \otimes R^1) \in \mathscr{LZ} \vee \mathscr{V} \subseteq \mathscr{U} \otimes (\mathscr{RZ} \otimes \mathscr{V}).$$

By Corollary 4.2, we also have

$$\begin{aligned} \mathrm{Reg}((G \otimes T) \otimes R^1) &\in \mathscr{LZ} \circ (\mathscr{U} \otimes (\mathscr{RZ} \otimes \mathscr{V})) \\ &= \mathscr{LZ} \circ \langle (\mathscr{LZ} \vee \mathscr{U}) \circ (\mathscr{RZ} \otimes \mathscr{V}) \rangle_{er} \quad \text{by Result 3.6} \\ &\subseteq \langle (\mathscr{LZ} \circ (\mathscr{LZ} \vee \mathscr{U})) \circ (\mathscr{RZ} \otimes \mathscr{V}) \rangle_{er} \quad \text{by Result 2.4} \\ &= \langle (\mathscr{LZ} \vee \mathscr{U}) \circ (\mathscr{RZ} \otimes \mathscr{V}) \rangle_{er} \\ &= \mathscr{U} \otimes (\mathscr{RZ} \otimes \mathscr{V}) \quad \text{by Result 3.6.} \end{aligned}$$

It follows that $\mathrm{Reg}(S \otimes R) \in \mathscr{U} \otimes (\mathscr{RZ} \otimes \mathscr{V})$. Hence $(\mathscr{U} \otimes \mathscr{RZ}) \otimes \mathscr{V} \subseteq \mathscr{U} \otimes (\mathscr{RZ} \otimes \mathscr{V})$.

(b) Let $\mathscr{V} \in \mathscr{L}_{er}(\mathscr{ES})$ or $\mathscr{L}_{er}(\mathscr{I}L)$. From (a), it remains to show that $\mathscr{U} \otimes (\mathscr{RZ} \otimes \mathscr{V}) \subseteq (\mathscr{U} \otimes \mathscr{RZ}) \otimes \mathscr{V}$. So let $G \in \mathscr{U}$ and $S \in \mathscr{RZ} \otimes \mathscr{V}$. By Result 3.5, we have $\mathscr{RZ} \otimes \mathscr{V} \subseteq \mathscr{ES}$ or $\mathscr{I}L$. Thus by Result 3.3, there exist $T \in \mathscr{RZ}$ and $R \in \mathscr{V}$ such that $S <_r \mathrm{Reg}(T \otimes R)$. Since $\mathscr{V}$ is monoidal, $R^1 \in \mathscr{V}$, and clearly $\mathrm{Reg}(T \otimes R) <_r \mathrm{Reg}(T \otimes R^1)$. It now follows from Lemmas 3.8 and 4.3 that

$$G \otimes S <_r G \otimes \mathrm{Reg}(T \otimes R) <_r G \otimes \mathrm{Reg}(T \otimes R^1) <_r \mathrm{Reg}((G \otimes T^{R^1}) \otimes R^1).$$

Since $\mathrm{Reg}((G \otimes T^{R^1}) \otimes R^1) \in (\mathscr{U} \otimes \mathscr{RZ}) \otimes \mathscr{V}$, we have $G \otimes S \in (\mathscr{U} \otimes \mathscr{RZ}) \otimes \mathscr{V}$. Hence $\mathscr{U} \otimes (\mathscr{RZ} \otimes \mathscr{V}) \subseteq (\mathscr{U} \otimes \mathscr{RZ}) \otimes \mathscr{V}$, as required. □

Corollary 6.11. *Let $\mathscr{U} \in \mathscr{L}_{ev}(\mathscr{CS})$ be such that $\mathscr{U} \cap \mathscr{G} \neq \mathscr{T}$, and $\mathscr{V} \in \mathscr{L}_{ev}(\mathscr{RS})$ be monoidal. Then*
(a) $\mathscr{U} \otimes \mathscr{V} \subseteq (\mathscr{U} \cap \mathscr{G}) \otimes (\mathscr{RZ} \otimes \mathscr{V})$;
(b) *in particular, if* $\mathscr{V} \subseteq \mathscr{ES}$ *or* $\mathscr{IL}$, $\mathscr{CS} \otimes \mathscr{V} = \mathscr{G} \otimes (\mathscr{RZ} \otimes \mathscr{V})$.

Proof. This follows immediately from Theorem 6.10 and the fact that $\mathscr{U} \subseteq (\mathscr{U} \cap \mathscr{G}) \otimes \mathscr{RZ}$ and $\mathscr{CS} = \mathscr{G} \otimes \mathscr{RZ}$. □

Our main result of this section is now an immediate consequence of the above results.

Theorem 6.12. *Let $\mathscr{V} \in \mathscr{L}_{ev}(\mathscr{RS})$ be a monoidal e-variety such that $\mathscr{RZ} \otimes \mathscr{V} = \mathscr{V}$. Then for any $\mathscr{U} \in \mathscr{L}_{ev}(\mathscr{CS})$, we have*

$$\mathscr{U} \otimes \mathscr{V} = \begin{cases} (\mathscr{U} \cap \mathscr{G}) \otimes \mathscr{V} & \text{if } \mathscr{U} \cap \mathscr{G} \neq \mathscr{T}, \\ (\mathscr{U} \cap \mathscr{LZ}) \vee \mathscr{V} & \text{otherwise.} \end{cases}$$

In particular, $\mathscr{CS} \otimes \mathscr{V} = \mathscr{V}^{T_l}$.

Corollary 6.13. (i) $\mathscr{CS} \otimes \mathscr{S} = (\mathscr{RZ} \vee \mathscr{S})^{T_l}$;
(ii) $\mathscr{CS} \otimes \mathscr{B} = \mathscr{B}^{T_l}$;
(iii) [12, Lemma 3.12] $\mathscr{CS} \otimes \mathscr{CR} = \mathscr{CR}$;
(iv) $\mathscr{CS} \otimes \mathscr{I} = (\mathscr{RZ} \vee \mathscr{I})^{T_l}$;
(v) $\mathscr{CS} \otimes \mathscr{C} = \mathscr{C}^{T_l}$;
(vi) [12, Corollary 3.13] $\mathscr{CS} \otimes \mathscr{ES} = \mathscr{ES}$.

Proof. This follows from Corollaries 6.3, 6.11 and Proposition 6.7. □

For completely regular semigroup *e*-varieties, we have the following general result.

Result 6.14 (Petrich and Reilly [22]). *Let $\mathscr{V} \in \mathscr{L}(\mathscr{CR})$ and $\mathscr{V} = [u_\alpha = v_\alpha]_{\alpha \in A}$. Then*

$$\mathscr{V}^{T_l} = [xu_\alpha = xu_\alpha(xv_\alpha)^0, xv_\alpha = xv_\alpha(xu_\alpha)^0]_{\alpha \in A},$$

where $x \notin c(u_\alpha) \cup c(v_\alpha)$ for all $\alpha \in A$.

Theorem 6.15. *Let $\mathscr{V} \in \mathscr{L}_{ev}(\mathscr{CR})$ with $\mathscr{RZ} \subseteq \mathscr{V}$. Then $\mathscr{CS} \otimes \mathscr{V} = \mathscr{V}^{T_l}$.*

Proof. If $\mathscr{V} \subseteq \mathscr{CS}$, then by [12, Proposition 3.10], $\mathscr{CS} \otimes \mathscr{V} = \mathscr{CS} = \mathscr{RZ}^{T_l} = \mathscr{V}^{T_l}$. So we assume that $\mathscr{S} \subseteq \mathscr{V}$ and $\mathscr{V} = [u_\alpha(x_i) = v_\alpha(x_i)]_{\alpha \in A}$. By Result 3.6, $\mathscr{V}^{T_l} = \mathscr{G} \otimes \mathscr{V} \subseteq \mathscr{CS} \otimes \mathscr{V}$. It remains to show that $\mathscr{CS} \otimes \mathscr{V} \subseteq \mathscr{V}^{T_l}$. Note that from Corollary 6.13 that $\mathscr{CS} \otimes \mathscr{V} \subseteq \mathscr{CR}$. By [36, Proposition 2.7] and Result 6.14, we have

$$\mathscr{V} = [u_\alpha(x_i)x = v_\alpha(x_i)x]_{\alpha \in A}$$

and

$$\mathscr{V}^{T_l} = [yu_\alpha x = yu_\alpha x(yv_\alpha x)^0, yv_\alpha x = yv_\alpha x(yu_\alpha x)^0]_{\alpha\in A},$$

where $x, y \notin c(u_\alpha)\cup c(v_\alpha)$ for all $\alpha\in A$. Now let $S\in\mathscr{C}\mathscr{S}$ and $T\in\mathscr{V}$, and let (f_i,t_i), $(f,t),(g,s)\in \operatorname{Reg}(S\otimes T)$. Since $T\in\mathscr{V}$, it follows that

$$\begin{aligned} &u_\alpha(t_i) = v_\alpha(t_i) = q, \text{ say,}\\ &(f,t)u_\alpha((f_i,t_i))(g,s) = (f,t)(f',u_\alpha(t_i))(g,s) \quad \text{for some } f'\in S^{T^1}\\ &\qquad\qquad = (f^t f'^{tq} g, tqs) \end{aligned} \tag{8}$$

and

$$\begin{aligned} &(f,t)v_\alpha((f_i,t_i))(g,s) = (f,t)(f'',v_\alpha(t_i))(g,s) \quad \text{for some } f''\in S^{T^1}\\ &\qquad\qquad = (f^t f''^{tq} g, tqs). \end{aligned} \tag{9}$$

Our next step is to show that the elements in (8) and (9) are $\mathscr{L}$-equivalent in $\operatorname{Reg}(S\otimes T)$. Towards this end, let $e=(tq)^0$ and define $h\in S^{T^1}$ by $xh=(xf)(xtf')(xtqg)[(xef)(xetf'')(xetqg)]^{-1}$. To see that $(h,e)\in\operatorname{Reg}(S\otimes T)$, let $k\in S^{T^1}$ be defined by $xk=(xeh)^{-1}$. Then

$$(h,e)(k,e)(h,e) = (h^e k^e h, e),$$

where, for all $x\in T^1$,

$$\begin{aligned} x(h^e k^e h) &= (xh)(xek)(xeh)\\ &= (xh)(xeh)^{-1}(xeh). \end{aligned}$$

Now, since S is completely simple,

$$xeh = (xef)(xetf')(xetqg)[(xef)(xetf'')(xetqg)]^{-1}$$

and

$$xh = (xf)(xtf')(xtqg)[(xef)(xetf'')(xetqg)]^{-1},$$

it follows that $xh\mathscr{L}(xetq)g\mathscr{L}xeh$. Hence $x(h^e k^e h)=xh$ for all $x\in T^1$ so that $(h,e)(k,e)(h,e)=(h,e)$. Therefore $(h,e)\in\operatorname{Reg}(S\otimes T)$. Now consider

$$\begin{aligned} (h,e)(f,t)v_\alpha((f_i,t_i))(g,s) &= (h,e)(f^t f''^{tq} g, tqs)\\ &= (h^e f^{et} f''^{etq} g, tqs). \end{aligned} \tag{10}$$

For any $x\in T^1$,

$$\begin{aligned} x(h^e f^{et} f''^{etq} g) &= (xh)(xef)(xetf'')(xetqg)\\ &= (xf)(xtf')(xtqg)[(xef)(xetf'')(xetqg)]^0 \end{aligned}$$

$$
\begin{aligned}
&= (xf)(xtf')(xtqg)[(xef)(xetf'')(xtqg)]^0 \\
&= (xf)(xtf')(xtqg) \\
&= x(f^t f'^{tq} g), \qquad (11)
\end{aligned}
$$

since $xf(xtf')(xtqg)\mathscr{L}xtqg\mathscr{L}(xef)(xetf'')(xtqg)$ in S. It follows from (8), (10) and (11) that $(h,e)(f,t)v_\alpha((f_i,t_i))(g,s) = (f,t)u_\alpha((f_i,t_i))(g,s)$. By symmetry, it follows that

$$(f,t)u_\alpha((f_i,t_i))(g,s)\ \mathscr{L}(f,t)v_\alpha((f_i,t_i))(g,s)$$

in $\mathrm{Reg}(S \otimes T)$. Hence,

$$(f,t)u_\alpha((f_i,t_i))(g,s) = (f,t)u_\alpha((f_i,t_i))(g,s)[(f,t)v_\alpha((f_i,t_i))(g,s)]^0$$

and

$$(f,t)v_\alpha((f_i,t_i))(g,s) = (f,t)v_\alpha((f_i,t_i))(g,s)[(f,t)u_\alpha((f_i,t_i))(g,s)]^0$$

from which it follows that the identities

$$yu_\alpha x = yu_\alpha x(yv_\alpha x)^0 \quad \text{and} \quad yv_\alpha x = yv_\alpha x(yu_\alpha x)^0$$

hold in $\mathrm{Reg}(S \otimes T)$. Consequently, $\mathrm{Reg}(S \otimes T) \in \mathcal{V}^{T_l}$ and therefore $\mathcal{CS} \otimes \mathcal{V} \subseteq \mathcal{V}^{T_l}$, as required. □

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